Polarity in Tetrahedral Complexes

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Introduction

Line Geometry, as the name suggests, is a study of the properties of figures formed of lines. In elementary projective geometry the set of coplanar lines in a point, or flat pencil, is well known, as well as the set of tangents to a conic section. Quadric surfaces are of the second order and class in that a line meets one in at most two points and at most two tangent planes lie in that line. In projective geometry there are three types: ellipsoids, ruled quadrics and imaginary quadrics. Only in affine geometry are hyperboloids distinguished from ellipsoids, and only in metric geometry are spheres distinct. Ruled quadrics are generally well known, and may be hyperboloids, paraboloids or cones. They are the next most basic form in line geometry.

The abbreviation “wrt” will be used for “with respect to”. Generally lower case latin symbols will be used for lines, upper case latin for points and Greek for planes and parameters. Exotic or italic capitals will be used for figures or configurations. When the term “general” is used, e.g. 6 general points, this means that special cases such as collinearity of three of them is excluded. Many standard results must be assumed to retain focus and brevity, which if not otherwise referenced will be found in S&K.

Line Geometry

In spatial geometry figures composed of planes or points are defined by means of a construction or an equation. Those with one degree of freedom possess $\infty^1$ points or planes forming a curve or a developable, while those with two degrees of freedom are surfaces. Three degrees of freedom? Not for points or planes. However there are $\infty^4$ lines in space, which is simply appreciated by considering all the lines joining the points of two planes. Thus there can be configurations with three degrees of freedom, which are known as line complexes. Configurations with two degrees of freedom are called congruences, while those with one are ruled quadrics. The set of tangents to a conic section is a degenerate case sometimes
called a *disc quadric*. The subject begins with the question: how are such figures of lines defined and what are their properties? The ruled quadric arises most simply as the lines of intersection of corresponding planes of two projective axial pencils with skew axes, or dually as the lines joining corresponding points of two projective ranges on two skew lines.

**Congruences**

Given two skew lines $p$ and $q$ the set of all lines intersecting them both forms the first type of linear congruence called the *hyperbolic congruence*. This is an example of a $(1,1)$ or linear congruence. It is so called because through any general point in space just one line can be drawn intersecting both $p$ and $q$, making the order of the congruence 1, while a general plane is met by $p$ and $q$ in two points $P$ and $Q$ which determine one line of the congruence, making its class 1 and thus $(1,1)$ overall. Through a familiar sequence, the congruence becomes a *parabolic strip* if the two lines coincide, determined by a projectivity between the planes and points of the line so that a pencil of the complex lies in a corresponding point/plane pair. Then an *elliptic congruence* arises when the two bearer lines become imaginary. Finally if the two bearer lines are real and intersect then a *degenerate congruence* arises consisting of all the lines in the plane and point of intersection of those lines. In the non-linear case many possibilities exist, including $(1,3)$ and $(2,2)$ congruences. Generally an $(m,n)$ congruence of order $m$ and class $n$ has $m$ lines in a general point and $n$ in a general plane, apart from singularities. A full treatment of general congruences may be found in Jessop.

An example of a $(1,3)$ congruence is the set of chords of a twisted cubic, for a plane contains 3 chords, and one passes through a general point (S&K). A selection of transversals is shown below.
(2,2) congruences may be generated from the transversals of three projective flat pencils $P_1 P_2 P_3$ two of which ($P_1$ and $P_2$) have a self-corresponding line (Jessop page 302).
This follows as any point $P$ projects two axial pencils in $P_1$ and $P_2$ which are in perspective and thus intersect in a flat pencil which meets the plane of $P_3$ in a line $p$. The resulting range on $p$ is projective with the range in which $p$ meets $P_3$, having two self-corresponding points each containing a line of the congruence. Dually a plane $\pi$ meets $P_1$ and $P_2$ in two perspective ranges determining a pencil which with $P_3$ determines two lines of the congruence as before. It is thus clear that the congruence contains $\infty^1$ reguli determined by the $\infty^1$ sets of three corresponding lines.

**Linear Complexes**

The simplest type of complex is the *special linear complex* consisting of all the lines intersecting a given axis $q$. It is clear that the lines of this complex lying in a general plane $\alpha$ form a flat pencil, since $q$ intersects $\alpha$ in a point which is the centre of a pencil of lines belonging to the complex. Dually, given a point $Q$ there is a flat pencil of lines of the complex in $Q$, for a plane $\alpha=(Q,q)$ contains a pencil centred on $Q$ which belongs to the complex. The *general linear complex* is also such that every point and plane of space contains a flat pencil of its lines, but there is no real axis. It may be generated by the lines intersecting corresponding pairs of lines of two projective flat pencils in distinct planes, provided those pencils possess a self-corresponding line (Sylvester's Theorem, c.f. Jessop page 46).

![Figure 1](image_url)

This shows the complex to be composed of hyperbolic congruences. Given a general plane $\alpha$ the two pencils intersect it in projective ranges which furthermore are perspective since the axis is self-corresponding, so the lines of the complex in $\alpha$ meeting corresponding points form a pencil. The dual case is clear, based on perspective axial pencils.
A linear complex meets a general line in a parabolic linear congruence since there is a flat pencil in every point and plane of the line.

Two linear complexes intersect in a linear congruence since each has a flat pencil in a point or plane, and two distinct pencils possess one common line.

Polarity is defined as follows.

![Figure 2](image)

Given a line $p$ not belonging to the complex, there is a flat pencil in each plane through $p$, and the centres of those pencils lie on a line $p'$ which is said to be the polar of $p$. For, given two planes $\alpha \beta$ each containing a pencil of the complex, join their centres to give $p'$. Take any plane $\gamma$ in $p$ and consider the two intersecting pencil rays $u$ and $v$. They determine a pencil of the complex with a line in $\gamma$ meeting $p'$. This is true of every such pair of rays intersecting on $p$, yielding a pencil of the complex in $\gamma$ centred on $p'$, which proves the result as $\gamma$ is general.

*Vice versa* the planes in $p'$ contain flat pencils with their centres on $p$, so polarity is reciprocal or *static* in the linear complex, unlike the case we will consider below. Obviously the lines of the complex itself are self-polar.

A point/plane polarity is defined by the fact that the flat pencil of the complex in a point determines a plane, and *vice versa*, which is known as a null polarity.
Tetrahedral Complexes

After linear complexes we come to quadratic complexes, which in general possess a disc quadric of lines in every plane of space and a quadratic cone in every point. An example is the special complex which consists of all the lines tangential to a quadric surface where it is easy to visualise the cone of its lines in a general point and the disc quadric in a plane that intersects the surface. Another more degenerate example consists of all the lines intersecting a conic, where all but one of the disc quadrics are pairs of flat pencils.

The tetrahedral complex is a quadratic complex which consists of all the lines intersecting the four planes of a given tetrahedron $T$ in a range with a fixed cross-ratio.

![Figure 3](image)

That those lines form a complex is plausible since there are $\infty^1$ possible cross-ratios so $\infty^3$ of the possible $\infty^4$ lines of space have been singled out. The most basic and important theorem is von Staudt's Theorem which states that the axial pencil of four planes in a line $p$ containing the vertices of $T$ has the same cross-ratio as the range in which $p$ meets the faces of $T$. This is readily proved explicitly (Appendix C) but is also clearly true by duality.

We will now examine the cross-ratio property of these complexes.
Given four lines in a plane, the set of lines intersecting them with a given fixed cross-ratio envelope a conic. For, given any four distinct lines $a, b, c, d$ no three of which form a pencil (Figure 4), choose any point $P$ on $d$, join $OP$ and find the line $OQ$ meeting $c$ in $Q$ such that the pencil $a,b,OP,OQ$ has cross-ratio $k$. Then the line $QP$ meets $a, b, c, d$ in a range with that cross-ratio. If we vary $P$ then the ranges $(P)$ and $(Q)$ are projective and thus the lines $QP$ envelope a conic. If $P$ lies on $d$ then $c$ is the line $QP$ so that $c$ is a tangent to the conic, and the same applies to $a, b$ and $d$.

Thus as $T$ intersects a general plane $\alpha$ in four lines, it follows that the lines of the complex in $\alpha$ envelope a conic. By the dual theorem the lines of the complex in a point form a quadratic cone, which confirms that we are dealing with a quadratic complex.

The complex may be generated using Sylvester's construction where the two pencils do not have a common line, for then the ranges in planes are projective yielding disc quadrics, and the pairs of axial pencils in points are projective intersecting in quadratic cones. There are two self-corresponding points on the line where the pencils intersect, which together with the pencil centres give the vertices of the tetrahedron. Two of its faces are readily seen to be total planes since each contains two corresponding rays.
The lines of the complex meeting a general line q not in the complex form a (2,2) congruence. For, an arbitrary point P determines with q a plane containing a disc quadric of complex lines, so there are at most two lines of that disc quadric in P which thus belong to the congruence. Dually an arbitrary plane $\alpha$ meets q in a point containing a cone of the complex, at most two lines of which lie in $\alpha$. Hence the congruence is (2,2).

The intersection of a tetrahedral and linear complex is a (2,2) congruence. For any point P contains a cone of the former and and a pencil of the latter, giving at most two common lines, and dually.

The intersection of two tetrahedral complexes is in general a (4,4) congruence, for any plane contains two disc quadrics which may share up to four lines, and dually for the two cones in a point.

A collineation of space with the tetrahedron as reference tetrahedron transforms all the lines of a tetrahedral complex into lines with the same cross-ratio, so the complex as a whole is invariant. A sequence of lines as the collineation is repeated lies in the developable surface of a path curve.

Complexes sharing the same tetrahedron are said to be cosingular complexes, for the tetrahedron is the singular surface of the complex. For, the planes of $\mathbf{T}$ are total planes, meaning that all lines lying in them belong to the complex. This
follows if we consider a disc quadric in a general plane \( \alpha \), as the lines in which \( \alpha \) meets the faces of \( T \) belong to that quadric since they are tangents to \( \alpha \), and hence belong to the complex. Every line of a face contains many such planes from which the result follows. Dually the vertices of \( T \) are total points.

It might seem that two cosingular complexes have a null intersection since their cross-ratios are distinct. However the lines in the total planes and points have singular cross-ratios and are obviously common to all the cosingular complexes.

For a given line of the complex, two cones sharing it intersect in a residual twisted cubic \( C \) circumscribed to the tetrahedron \( T \). In a point \( P \) of \( C \) there are two lines of the complex belonging to the above cones which are chords of \( C \), together with four other chords through the vertices of \( T \), which between them determine a unique cone \( \zeta \). There is also a cone of the complex with \( P \) as vertex which must be \( \zeta \) since it shares the above lines determining it. Hence all the chords of \( \zeta \) belong to the complex, and they form a \((1,3)\) congruence (S&K). There are \( \infty^3 \) lines of the complex and \( \infty^2 \) pairs of cones sharing each one, apparently giving \( \infty^5 \) circumscribed twisted cubics. But since there are \( \infty^1 \) chords in \( P \) belonging to the complex, each one together with those through the vertices determines a cone of the complex, so there are \( \infty^2 \) pairs of cones of the complex forming a pencil in \( C \) and hence \( \infty^2 \) duplications. This leaves \( \infty^3 \) distinct circumscribed twisted cubics and thus \( \infty^3 (1,3) \) congruences of the complex. This reasoning applies to all the cosingular complexes so there are \( \infty^4 \) circumscribed twisted cubics in all. Since 6 general points determine a twisted cubic, two points may be chosen in addition to the vertices giving \( \infty^6 \) cubics, but there are \( \infty^2 \) duplications since there are \( \infty^2 \) ways of choosing two points on a curve, yielding \( \infty^4 \) circumscribed cubics in all, which keeps the book-keeping straight!

Another way of generating the above \((1,3)\) congruences is to establish a collineation between the bundles of lines in two of the vertices of \( T \), for which intersecting corresponding lines meet in a twisted cubic \( C \) (S&K). The collineation must have corresponding lines intersecting in the other two vertices for \( C \) to circumscribe \( T \). Since four corresponding pairs determine such a collineation, there are \( \infty^2 \) choices for each of the other two pairs and hence \( \infty^4 \) collineations in all.
**Polarity**

Polarity in a quadratic complex is approached in a similar way to that in a linear complex.

![Figure 5](image)

Given a line $p_1$ whose polar we seek (Figure 5), we know that in each plane of the axial pencil through $p_1$ there is a disc quadric, and furthermore there is a point $P$ polar to $p_1$ with respect to the conic enveloped by that disc quadric. The points such as $P$ in the axial pencil lie on a line $p_2$ which is said to be the polar of $p_1$. For, consider the tangents $u$ and $v$ in $O$ as shown. In the other plane we have $u_1$ and $v_1$ in $O$, and these tangents must all belong to a quadratic cone $\mathcal{C}$, as must all such tangents in all the planes of the axial pencil in $p_1$. There is a line/plane polarity in the point $O$ wrt $\mathcal{C}$ for which the polar of $p_1$ is a plane $\pi$ harmonically related to $p_1$. Now the rays $OP$ and $p_1$ separate $u$ and $v$ harmonically. and $\pi$ must meet the plane $(u,v)$ in a line harmonic to $p_1$ wrt $u$ and $v$ (which lie in the cone), so $\pi$ meets the plane $(u,v)$ in the line $OP$. Similarly it meets the plane $(u_1,v_1)$ in the line $OP_1$, so $P$ and $P_1$ lie in $\pi$. The same applies to the cone in any other general point on $p_1$, yielding another plane $\pi_1$ which also contains the points $P$ and $P_1$. Since $\pi$ and $\pi_1$ meet in a line the result follows.
In the case of a complex of lines touching a quadric surface the polarity is clearly the same as the ordinary point/plane one, and like a linear complex it is reciprocal, the polar of $p_2$ being $p_1$.

For a tetrahedral complex $K$, if we repeat the whole process for $p_2$ we obtain in general a third line $p_3$ as the polar of $p_2$; we do not return to $p_1$. Thus polarity is in a sense *fluid* unlike the static situation above. We may repeat this process to obtain the *polar descendents* of $p_1$, which raises the question: are they rulers of a figure or surface?

To answer this, we need some subsidiary results. If we take a plane $\alpha$ containing a vertex $V_1$ of $T$ (see Figure 6 below) then there is a flat pencil of the complex in $V_1$ (as $V_1$ is a total point), so the remainder of the disc quadric $Q$ in $\alpha$ must be another flat pencil $P$. If $\alpha$ meets the face $\nu$ opposite $V_1$ in a line $k_1$ then $k_1$ is a line of the complex since $\nu$ is a total plane, so the centre $K_1$ of $P$ must lie on $k_1$. If the line of $P$ through $V_1$ meets a line $x$ in $L_1$ then the polar of $x$ must pass through the point $M_1$ such that $L_1M_1$ harmonically separate $K_1V_1$. This follows because we know from basic projective geometry that a line $u$ through the pole $P$ of a line $p$ wrt a conic intersects that conic in points harmonically separating $P$ and $(u,p)$, and the result for the degenerate two-pencil case follows immediately.

For a given complex there are four planes through $x$ containing the vertices and hence four lines such as $K_1L_1$, and four mutually skew lines possess in general two transversals. $x$ is one of the transversals in this case, and since the polar line of $x$ meets them all, it is the other one. This gives us another more serviceable way of finding the polar of a line, which will be important later.
Referring to the above figure and what went before, we wish to construct the polar line of the line $x$. We saw that the polar line $x'$ of $x$ intersects $K_1V_1$ in the point $M_1$ harmonic to $L_1$ wrt $K_1$ and $V_1$. As we traverse the cosingular complexes $K_1$ describes $k_1$, and $M_1$ thus describes a line $m_1$ harmonic to $x$ wrt $k_1$ and $V_1$. Hence the polars of $x$ wrt the cosingular complexes all intersect $m_1$. If we repeat this for the planes $\alpha$ determined by $x$ and the other vertices $V_j$ then we obtain four lines $m_j$ which are met by all the polars of $x$. Since $x$ intersects the $m_j$ where it meets the $k_j$ and is skew to its polars, the $m_j$ must be skew and hence the polars of $x$ wrt the cosingular complexes form a regulus $\mathcal{R}$. Evidently $x$ also belongs to $\mathcal{R}$ since it intersects the $m_j$. It follows that for any cosingular complex the polar of $x$ belongs to $\mathcal{R}$.

From the above harmonic considerations it follows that the quadric surface $Q$ containing $\mathcal{R}$ is such that the vertices of the tetrahedron and their opposite faces are harmonic wrt it, so the tetrahedron is *self-polar* wrt $Q$ (we use italics to indicate a point/plane polarity in contrast to the line/line polarity in the complex). For brevity we will refer to such quadrics as *self-polar* quadrics, although in reality it is the tetrahedron that is *self-polar* wrt them. Now there is a unique *self-polar* quadric through a given line, since that line determines four harmonically related lines wrt the tetrahedron as we have seen, all of which lie on the quadric and so determine it uniquely (three skew lines suffice to determine a ruled quadric). Thus if we take any ruler $r$ of $\mathcal{R}$, then $\mathcal{R}$ is itself the unique *self-polar* quadric determined by $r$, and the polars to $r$ wrt the cosingular complexes lie on $\mathcal{R}$ as already proved. Thus for any cosingular complex, the polar line of every ruler of $\mathcal{R}$ belongs to $\mathcal{R}$, so the latter is
non-trivially self-polar wrt that complex. Consequently all the polar descendents of any ruler of $R$ must lie on $R$. Although this applies to all of the cosingular tetrahedral complexes, the polar descendents of a line $p$ wrt to one complex are not the same as those wrt to another, even though both sets lie on $R$.

To sum up, we have the strikingly simple result that the basic self-polar figures wrt a tetrahedral complex are the reguli for which the tetrahedron is self polar. A deeper result is that the self-polar reguli are the same for all cosingular complexes.

Other non-trivially self-polar figures are composed of all or parts of such reguli, and include congruences and scrolls as well as linear and tetrahedral complexes.

**Algebraic Approach**

We now give the algebraic approach to the above assuming the reader is familiar with homogeneous coordinates, and in particular the Plücker line coordinates $p_{jk}$. Recall that for the line $p_{jk}$ joining two points $x_j$ and $y_j$, $p_{jk} = x_jy_k - x_ky_j$, there being six such coordinates that are independent of the selection of the two points $x_j$ and $y_j$ on the line. Given any six numbers $p_{jk}$ they only represent a line if they satisfy the Plücker line condition

$$ p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0 \quad (1) $$

and two lines $p_{jk}$ and $q_{jk}$ intersect if

$$ p_{01}q_{23} + p_{02}q_{31} + p_{03}q_{12} + p_{12}q_{03} + p_{31}q_{02} + p_{23}q_{01} = 0 \quad (2) $$

We take the tetrahedron as the reference tetrahedron for our coordinates.

The equation of a general linear complex is $\Sigma a_{ij}p_{kl}=0$ where summation occurs for all combinations of $i \neq j \neq k \neq l$ as in (2), $a_{ij}$ being six arbitrary constants and $p_{kl}$ are the coordinates of any line belonging to the complex. For an arbitrary fixed point $y_j$ we have $A_0x_0 + A_1x_1 + A_2x_2 + A_3x_3 = 0$ where e.g. $A_0 = a_{23}y_1 + a_{31}y_2 + a_{12}y_3$ i.e. all points on lines of the complex through $y_j$ lie in the plane $(A_0, A_1, A_2, A_3)$, giving a flat pencil. A dual argument gives a pencil in a plane, so the complex is indeed linear. It is special if the $a_{ij}$ satisfy the Plücker line condition in which case they are the coordinates of the real bearer line since the equation is the same as (2).

The equation of a tetrahedral complex with cross-ratio $k$ is (S&K)
\[ p_{02}p_{31} + kp_{03}p_{12} = 0 \]  \hspace{1cm} (3)

For, consider a line \( p \) given by \((x_j - \lambda y_j)\) where \( \lambda \) is a parameter. \( p \) meets the faces of the tetrahedron when \( \lambda_j = x_j / y_j \) as \( x_j = 0 \), so the cross-ratio \( k \) of those points is

\[
k = \left\{ \frac{x_0}{y_0}, \frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{x_3}{y_3} \right\} = \frac{x_0 y_2 - x_2 y_0}{x_1 y_2 - x_2 y_1} \div \frac{x_0 y_3 - x_3 y_0}{x_1 y_3 - x_3 y_1} = -\frac{p_{02}p_{31}}{p_{03}p_{12}}
\]

For the lines of the complex meeting an edge of the tetrahedron one of the \( p_{ij} \) is zero, say \( p_{01} = 0 \) for the edge \( \{0 \ 0 \ 0 \ 0 \ 1\} \). The line condition (1) then gives \( p_{02}p_{31} + p_{03}p_{12} = 0 \) so we have \( \{0 \ p_{02} \ p_{03} \ p_{12} / p_{02} \ p_{23}\} \). Applying (3) shows that either \( p_{03} \) or \( p_{12} \) is zero, leaving us with \( \{0 \ p_{02} \ p_{03} \ 0 \ 0 \ p_{23}\} \) or \( \{0 \ p_{02} \ 0 \ p_{12} \ 0 \ p_{23}\} \). These are the lines in the plane \( \{0 \ 1 \ 0 \ 0\} \) and the vertex \( \{0 \ 0 \ 1 \ 0\} \) respectively, which is a degenerate linear congruence.

The line \( q_{ij} \) polar to a line \( p_{ij} \) of a tetrahedral complex is derived in Appendix F as follows:

\[
\begin{align*}
q_{01} &= Ap_{01} \\
q_{02} &= Bp_{02} \\
q_{03} &= Cp_{03} \\
q_{12} &= Cp_{12} \\
q_{31} &= Bp_{31} \\
q_{23} &= Ap_{23}
\end{align*}
\]

where

\[
\begin{align*}
A &= p_{02}p_{31} + k^2 p_{03}p_{12} \\
B &= p_{01}p_{23} + (1-k)^2 p_{03}p_{12} \\
C &= k^2 p_{01}p_{23} + (1-k)^2 p_{02}p_{31}
\end{align*}
\]

This exhibits a remarkable symmetry from which much follows. First of all a figure composed of the lines

\[
\{p_{01} \ p_{02} \ p_{03} \ dp_{03} \ cp_{02} \ bp_{01}\}
\]

is self-polar since its polar lines are \( \{Ap_{01} \ Bp_{02} \ Cp_{03} \ Cdp_{03} \ Bcp_{02} \ Abp_{01}\} = \{q_{01} \ q_{02} \ q_{03} \ dq_{03} \ cq_{02} \ bq_{01}\} \) i.e. they belong to the same figure. If \( b \ c \) and \( d \) are all constant then we have a self-polar regulus as there are two free ratios of coordinates after satisfying (1). Note that \( b \ c \) and \( d \) cannot all have the same sign or (1) cannot be satisfied by real numbers. It can be shown (Appendix A) that these lines lie in a self-polar quadric surface given by
\[ bcdx_0^2 + bx_1^2 + cx_2^2 + dx_3^2 = 0 \]  

(7)

If \( b, c \) or \( d \) varies then \( p_{23} \), \( p_{31} \) or \( p_{12} \) is independent and we have \( \infty^1 \) self-polar reguli forming a self-polar congruence, and if only one of \( b, c \) or \( d \) is fixed then we have a self-polar complex. For example if \( b \) is constant then these lines satisfy an equation 
\[ a_{01}p_{23} + a_{23}p_{01} = 0 \]
where \( b = -a_{23}/a_{01} \), so we see that the complex is linear, and similarly for constant \( c \) or \( d \). The special linear complexes such as \( p_{01} = 0 \) are clearly self-polar.

Returning to self-polar congruences, their nature depends upon the signs of \( b \) and \( c \) in (6), taking \( d \) to be variable. If \( b \) and \( c \) are of the same sign then \( d \) is of opposite sign and cannot change sign. Two of the reguli determined by \( d_1 \) and \( d_2 \) then have self-polar quadrics intersecting in 
\[ (d_1 - d_2)(bcx_0^2 + x_3^2) = 0 \]
using (7). They possess no real common points so the base quartic of the pencil is imaginary and hence the congruence is elliptic. If \( b \) and \( c \) are of opposite sign then \( bcx_0^2 + x_3^2 = 0 \) has real solutions giving two planes. Two other planes arise from 
\[ bx_1^2 + cx_2^2 = 0, \]
so the quadrics meet in a skew quadrilateral and hence the congruence is hyperbolic.

Now consider the congruence \((d \ 0 \ b)\) with \( c = 0 \). Its self-polar quadrics are 
\[ bx_1^2 + dx_3^2 = 0 \]
as \( d \) varies, which are pairs of real planes since \( b \) and \( d \) must be of opposite sign. The self-polar reguli thus degenerate to pairs of flat pencils, yielding a parabolic congruence in the edge \( \{0 \ 1 \ 0 \ 0 \ 0 \ 0 \} \). Finally if \( b = c = 0 \) then we have the plane 
\[ \{0 \ 0 \ 0 \ 1\} \] as \( x_3 = 0 \), giving a degenerate congruence. The above congruences are pencils of the form \( S_1 + \mu S_2 = 0 \), where \( S_1 \) is given by \( d_1 \) and \( S_2 \) by \( d_2 \). We may vary \( b \) or \( c \) instead to give all cases.

To summarise for variable \( d \):

<table>
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<th>congruence</th>
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<td>degenerate</td>
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Table 1

A more detailed treatment is given in Appendix E.

The polar of a cosingular tetrahedral complex is, using (3) and (4), 
\(B^2p_{02}p_{31}+\lambda C^2p_{03}p_{12}=0\), which is also a cosingular complex with cross-ratio \(\lambda C^2/B^2\). 
\(B=C\) only if \(k=1\) or \(\lambda=k\), so there are no self-polar cosingular complexes.

Now consider the self-polar complex given by \(\{p_{01}, p_{02}, p_{03}, -\mu p_{02}/g, p_{01}/(\mu g)\}\) 
with \(g\) constant and \(\mu\) a parameter. For a given value of \(\mu\) we have a self-polar hyperbolic congruence, and as \(\mu\) varies we see that the complex is composed of such congruences. The self-polar quadrics are 
\(d(x_0/g)^2+x_1^2/(\mu g)-\mu x_2^2/g+dx_3^2=0\) and when \(\mu=0\) we have the plane \(x_1=0\) i.e. a face of the tetrahedron. Similarly when \(\mu=\infty\) we have \(x_2=0\). When \(d=0\) we have \(x_1^2=(\mu x_2)^2\) giving two harmonically separated total planes. Thus there are four total planes so the complex is tetrahedral. As \(g\) varies we obtain \(\infty^1\) such complexes. Hence there exist self-polar tetrahedral complexes, the other two sets of this type being given by 
\(\{p_{01}, p_{02}, p_{03}, -\mu p_{03}/g, p_{02}/(\mu g), p_{23}\}\) and \(\{p_{01}, p_{02}, p_{03}, \mu p_{02}/g, p_{31}, -p_{01}/(\mu g)\}\).

Now let us find the intersection of two self-polar tetrahedral complexes, say 
\(\{p_{01}, p_{02}, p_{03}, -\mu p_{02}/g, p_{01}/(\mu g)\}\) and \(\{p_{01}, p_{02}, p_{03}, \sigma p_{02}/h, p_{31}, -p_{01}/(\sigma h)\}\). For common lines \(\sigma h=-\mu g\) so \(\sigma=\mu g/h\), giving the congruence 
\(\{p_{01}, p_{02}, p_{03}, -\mu gp_{02}/h^2, -\mu p_{02}/g, p_{01}/(\mu g)\}\) for which the self-polar quadric surfaces are 
\(g\mu^2x_0^2+h^2gx_1^2-h^2g\mu^2x_2^2-g^2\mu^2x_3^2=0\). Every point containing a line of the congruence must lie in one of these surfaces. Substituting for the coordinates of a 
given point on the surface we obtain two equal and opposite values of \(\mu\) for that point, so two generators pass through it, and we find the same state of affairs for the class 
quadric. Since two lines pass through every point on the self-polar quadric surfaces 
the congruence is composed of two sets of conjugate reguli. For two reguli to intersect we require 
\(\mu_1^2(gx_0^2-h^2gx_2^2-g^2x_3^2)\)=\(\mu_2^2(gx_0^2-h^2gx_2^2-g^2x_3^2)\) which is only possible for 
\(\mu_1=-\mu_2\) i.e. for the conjugate reguli. Thus apart from conjugates the reguli are all skew. The congruence is thus \(2,2\) and self-polar (c.f. the general case where it is \(4,4\)). The congruence does not degenerate to the sum of two linear congruences 
since for some points there is no real solution for \(\mu\).

The \((1,3)\) congruences belonging to the complex contain reguli circumscribed to the 
tetrahedron which are thus not self-polar and hence not self-polar. Thus these 
congruences cannot consist solely of self-polar reguli as the latter do not circumscribe 
the tetrahedron.
The lines polar to lines intersecting an edge of the tetrahedron also intersect that edge e.g. for the edge \( \{1,0,0,0,0\} \) the lines \( \{p_{01} p_{02} p_{12} p_{31} 0\} \) intersect it, and their polars are \( \{A p_{01} B p_{02} C p_{03} C p_{12} B p_{31} 0\} \). Thus the hyperbolic congruences in pairs of skew edges are self-polar. The polarity is reciprocal, for lines intersecting e.g. \( \{1,0,0,0,0\} \) and \( \{0,0,0,0,1\} \) are \( \{0 p_{02} p_{03} p_{12} p_{31} 0\} \) for which from (5) \( A=p_{02} p_{31}+k^2 p_{03} p_{12}, B=(1-k)^2 p_{02} p_{31} \) and since from (1) \( p_{02} p_{31}=p_{03} p_{12} \) we cancel the common factor to give \( A=(1+k)/(1-k), B=-1, C=1 \), so the polar lines are \( \{0 p_{02} p_{03} p_{12} p_{31} 0\} \).

There exist self-polar scrolls of lines i.e. on surfaces other than reguli. Consider for example the scroll \( \{\sigma^n, f\sigma^m, 0, 0, g\sigma^{n-m}, -fg\} \) where \( \sigma \) is the parameter and \( f \) and \( g \) are constants. From (5) \( A=fg\sigma^n, B=-fg\sigma^n \) and \( C=-k^2fg\sigma^n+(1-k)^2fg\sigma^n \). Removing the common factor we have \( A=1, B=-1, C=1-2k \). The polar lines are thus \( \{\sigma^n, -f\sigma^m, 0, 0, -g\sigma^{n-m}, -fg\} \). Repeating we arrive back at the original line, so we have a reciprocal polarity. For a polar line to belong to the figure, since \(-fg\) is constant \( n \) must be even to include negative values of \( \sigma \), and \( m \) must be odd to reverse the signs of \( \sigma^m \) and \( \sigma^{n-m} \). The scroll then contains all its own polars. All the lines meet the edges \( \{0,0,0,1,0,0\} \) and \( \{0,0,0,0,1,0\} \) which are the only directrices of these scrolls. We must have \( n=m \) for the \( p_{31} \) term, so the scroll is of even order. An example of an odd order scroll is \( \{\sigma^n, f\sigma^{m+1}, 0, 0, g\sigma^{n-m}, -fg\sigma\} \) where both \( n \) and \( m \) are odd. This gives \( A=-1 \) and \( B=1 \) so that the polar line is \( \{-\sigma^n, f\sigma^{m+1}, 0, 0, g\sigma^{n-m}, fg\sigma\} \) and thus the scroll contains its own polars. These are two easy cases where the Plücker condition (1) ensures that (5) gives constant \( A \) and \( B \), one negative. A self-polar cubic scroll of this type would then be \( \{\sigma^3, f\sigma^2, 0, 0, g\sigma^2, -fg\sigma\} \). This is the join of \( p=\{\sigma^3, 0, 0, g\sigma\} \) and \( q=\{0, \sigma, f, 0\} \), so for a point \( q \), \( p \) is uniquely determined. However, there are two points \( q \) corresponding to a given point \( p \) as \( \{\sigma^3, 0, 0, g\sigma\} \equiv \{-\sigma^3, 0, 0, -g\sigma\} \) gives two values of \( \sigma \) of opposite sign. The scroll is thus determined by a (1,2) correspondence between the lines \([0,0,0,1,0,0]\) and \([0,0,1,0,0,0]\).

**Polar Figures**

**Flat Pencil**

First we will consider the polar of a general flat pencil. Given two fixed lines \( p_{ij} \) and \( q_{ij} \) of the pencil, the latter is given by \( p_{ij} + \mu q_{ij} \). The polar of this is obtained using (4) and (5), giving

\[
A = (p_{02} + \mu q_{02})(p_{31} + \mu q_{31}) + k^2(p_{03} + \mu q_{03})(p_{12} + \mu q_{12})
\]
\[(p_{02}p_{31}+k^2p_{03}p_{12})+\mu^2(q_{02}q_{31}+k^2q_{03}q_{12})+\mu\{q_{02}p_{31}+p_{02}q_{31}+k^2(q_{03}p_{12}+p_{03}q_{12})\}\]
\[=A_p+\mu^2A_q+\mu A'\]

(9)

defining \(A_p\), \(A_q\), \(A'\) accordingly. Now there are two self-polar lines in the pencil which may be chosen as \(p_{ij}\) and \(q_{ij}\), in which case \(A_p=A_q=1\), so
\[A = 1+\mu^2+\mu A'.\]

Similarly
\[B = 1+\mu^2+\mu B'\]
\[C = 1+\mu^2+\mu C'\]

If the polar line is \(r_{ij}\), then \(r_{0i}=(1+\mu^2+\mu A')(p_{01}+\mu q_{01})\) which is cubic in \(\mu\). Similarly for the other coordinates of \(r_{ij}\), and thus the polar figure is a cubic scroll. This result is given by Jessop (page 171), although he proved it using Klein coordinates. Noting that there is a line \(d\) polar to the centre \(C\) of the pencil wrt the disc quadric of the complex in the plane of the pencil, every line of the scroll must meet \(d\), which is thus a directrix.

![Figure 7](image)

More generally, suppose a line \(s_{ij}\) intersects all the lines of the scroll. From (2)
\[s_{01}(1+\mu^2+\mu A')(p_{23}+\mu q_{23})+s_{02}(1+\mu^2+\mu B')(p_{31}+\mu q_{31})+...=0\]

which must be satisfied for all \(\mu\), so \(s_{ij}\) must intersect \(p_{ij}\) when \(\mu=0\) and \(q_{ij}\) when \(\mu=\infty\) (for the \(\mu^3\) equation). Since all rulers of the scroll intersect \(d_{ij}\), \(s_{ij}\) (\(\neq d_{ij}\)) can only intersect both \(p_{ij}\) and \(q_{ij}\) if it passes through the centre \(C\) of the pencil and does not lie in the plane of the pencil. The above equation is of the form
\[(1+\mu^2)\Sigma s_{ij}p_{kl} + (\mu+\mu^3)\Sigma s_{ij}q_{kl} + \mu(A's_{01}p_{23} + ...) + \mu^2(A's_{01}q_{23} + ...) = 0\]

i.e. \[(A's_{01}p_{23} + B's_{02}p_{31} + ...) + \mu(A's_{01}q_{23} + B's_{02}q_{31} + ...) = 0\]

as \(p_{ij}\) and \(q_{ij}\) intersect \(s_{ij}\). Again this must be independent of \(\mu\), so the two terms in braces must vanish independently. There are thus 4 equations in \(s_{ij}\), including the incidence conditions with \(p_{ij}\) and \(q_{ij}\), which together with the Plücker line condition (1) yield one solution for homogeneous coordinates. This gives a second directrix of the scroll passing through \(C\). If now we consider the cone \(C\) of the complex in \(C\), there exists a line \(u\) through \(C\) polar to the plane of the pencil wrt \(C\). For a line \(x\) of the pencil, the line \(x'\) of the scroll polar to \(x\) must intersect \(u\) as it lies in the plane polar to \(x\) wrt \(C\) (which contains \(u\)). Hence \(u\) is the second directrix just discovered. A cubic scroll is generated by a (1,2) correspondence between the points of two skew lines (S&K), which thus applies to \(u\) and \(d\). This is seen as follows.

![Figure 8](image-url)
The line $d$ is shown with the cone of the complex in $C$ and the two lines of the complex intersecting $d$. Given a line $x$ of the pencil through $X$ on $d$, its polar $x'$ passes through $Y$ on $d$ harmonic to $X$ wrt the cone. Let $x'$ meet the line $u$ in $X'$. Now $x$ determines a *self-polar* quadric, which contains $x'$ and the three points $X$, $Y$, and $C$. The line $y$ of the pencil through $Y$ has its polar line $y'$ passing through $X$. Thus $X$, $Y$, and $C$ lie on the *self-polar* quadric containing $y$ and $y'$. But these two quadrics meet in the coplanar lines $x$ and $y$ and thus possess a common conic through $X$ and $Y$. Since (7) has no cross-products of the coordinates that conic is a line pair (c.f. Todd page 190), i.e. the lines $x'$ and $y'$. Hence $y'$ must meet $u$ in $X'$, so every point on $u$ that lies on the scroll corresponds to two points on $d$. An example, of course, is $C$ containing the two self-polar lines.

Finally, note that if $q_{ij}$ is a singular line containing a vertex of $T$ then $A_p=0$ in (9) as three of its coordinates are zero, and the scroll is then quadratic i.e. a *regulus*. If both $p_{ij}$ and $q_{ij}$ contain distinct vertices then the plane $\alpha$ of the pencil contains an edge of $T$ and thus its polar lies in the plane harmonic to $\alpha$ wrt $T$ (see Appendix B). The scroll thus degenerates to a pencil. Thus the polars of flat pencils in planes containing one vertex are reguli, and those of pencils in planes containing an edge of $T$ are flat pencils.

**Plane**

It follows that the polar of a plane $\alpha$ of lines is a congruence $C_p$ composed of cubic scrolls. We note that for the pencil above, $u$ is the polar line of $d$ (but not *vice versa*). Given an arbitrary point $P$ in space, if it lies on a line such as $u$ through the centre of a pencil in $\alpha$ then there are the two lines of the congruence through $P$ due to the (1,2) correspondence. In addition $u$ belongs to $C_p$ since it is the polar of $d$, a line of $\alpha$. Thus the congruence is of order 3. An arbitrary plane $\pi$ intersects $\alpha$ in a line $d$, and two lines of $C_p$ may lie in $\pi$ should it happen to contain two lines of the scroll related to $d$. $C_p$ is thus a (3,2) congruence. $\alpha$ is of course singular as it contains a disc quadric of self-polar lines of $C_p$, and its points are singular as they contain cones of $C_p$.

Clearly two such congruences meet in general in one line i.e. are almost disjoint. Since $C_p$ contains $\infty^2$ scrolls each of its lines must contain a pencil of scrolls.

If $\alpha$ contains one vertex of $T$ then the congruence is composed of reguli, while if it contains an edge then it is the plane of lines harmonic to $\alpha$ wrt $T$ (Appendix B).

**Regulus**
A regulus $R$ determined by three mutually skew rulers $p_{ij}$, $q_{ij}$ and $s_{ij}$ is $r_{ij} = \lambda p_{ij} + \mu q_{ij} + \nu s_{ij}$ (S&K page 370). Thus for its polar:

$$A = (\lambda p_{02} + \mu q_{02} + \nu s_{02}) (\lambda p_{31} + \mu q_{31} + \nu s_{31}) + k^2 (\lambda p_{03} + \mu q_{03} + \nu s_{03}) (\lambda p_{12} + \mu q_{12} + \nu s_{12})$$

$$= \lambda^2 A_p + \mu^2 A_q + \nu^2 A_s + \lambda \mu (A_{pq} + A_{qp}) + \lambda \nu (A_{ps} + A_{sp}) + \mu \nu (A_{qs} + A_{sp})$$

where

$$A_p = p_{02} p_{31} + k^2 p_{03} p_{12}$$

and similarly for $A_q$ and $A_s$ which are constants,

$$A_{pq} = p_{02} q_{31} + k^2 p_{03} q_{12}$$

etc. which are constants for all lines $r_{ij}$.

When $A$ is multiplied by $r_{01}$ and $r_{23}$ (c.f. (4)) we obtain cubic terms in $\lambda \mu$ and $\nu$. Similarly for $B$ and $C$. Hence we obtain a cubic scroll. We saw earlier that a cubic scroll possesses two directrices, so for such a line $t_{ij}$ to intersect all its rulers, (2) gives

$$t_{01} \left( \lambda^2 A_p + \mu^2 A_q + \nu^2 A_s + \lambda \mu (A_{pq} + A_{qp}) + \lambda \nu (A_{ps} + A_{sp}) + \mu \nu (A_{qs} + A_{sp}) \right) \left( \lambda p_{23} + \mu q_{23} + \nu s_{23} \right) + ... = 0$$

This must hold for all $\lambda \mu$ and $\nu$. It consists of terms multiplying all cubic permutations of $\lambda \mu$ and $\nu$, of which there are 10, and the 10 terms multiplying those permutations must vanish independently. The $\lambda^3$ term shows that $t_{ij}$ must intersect the polar of $p_{ij}$, and similarly for $\mu^3$ and $\nu^3$ it must intersect the polars of $q_{ij}$ and $s_{ij}$. For a line of $R$ to be self-polar we require $A = B = C$ which gives two equations in $\lambda \mu$ and $\nu$, which must yield a porism in one of the latter so that either all lines are self-polar (which is possible), or else two are since the porism must effectively reduce the cubic residual to a quadratic equation. We then choose $q_{ij}$ and $s_{ij}$ as the self-polar lines of $R$ in which case $t_{ij}$ intersects them, so the scroll and $R$ possess two common self-polar lines which are intersected by the directrices.

If $R$ lies in a cosingular complex then its polar is also a regulus since $A, B$ and $C$ are the same constants for all its lines (c.f. (8)).

**Linear Complex**

Now consider the general linear complex $\Sigma a_{ij} p_{kl} = 0$. Its polar is

$$a_{01} A_{p_{23}} + a_{02} B_{p_{31}} + ... = 0$$

i.e.

$$a_{01} p_{23} \{ p_{02} p_{31} + k^2 p_{03} p_{12} \} + a_{02} p_{31} \{ p_{01} p_{23} + (1-k)^2 p_{03} p_{12} \} + ... = 0$$
which is a cubic complex. If the linear complex lies in a cosingular tetrahedral complex then we use (8) for A, B and C:

\[ a_{01}p_{23}\{k^2-\lambda}\}+a_{02}p_{31}\{k^2-2k+\lambda}\}+\ldots=0 \]

which is again a linear complex since k and \( \lambda \) are constant for all component lines. Clearly the polar of a cosingular complex is another cosingular complex.

**Cosingular Complexes**

This illustrates the general principle that the polars of figures in cosingular tetrahedral complexes are of the same type as the originals.

**General Polar Figures**

Otherwise the degree of the polar figure is in general three times that of the original, as was illustrated by the polar of a flat pencil. This is because A, B and C contain products of the line coordinates (c.f. (5)), and they multiply the coordinates of the original figure as in (4).

**Semi-Imaginary Tetrahedron**

The study of tetrahedra with some imaginary elements is derived in a similar way, but the coordinates are complex numbers. For a tetrahedron that is not fully real we retain a real reference tetrahedron which shares two skew edges with that of the complex, and for the results below those edges are \{1 0 0 0 0 0\} and \{0 0 0 0 0 1\}. For a semi-imaginary tetrahedron, where the real elements are two skew edges, two real planes and two real vertices, the expressions for polar lines are more complicated and turn out to be

\[
\begin{align*}
q_{01} &= Ap_{01} \\
q_{02} &= Bp_{02}+ Cp_{12} \\
q_{03} &= Bp_{03}+ Cp_{31} \\
q_{12} &= Bp_{12}+ Cp_{02} \\
q_{31} &= Bp_{31}- Cp_{03} \\
q_{23} &= Ap_{23}
\end{align*}
\]

where

\[
\begin{align*}
A &= (p_{02}p_{03}+p_{12}p_{31})\sin\theta + p_{01}p_{23}\cos\theta \\
B &= -p_{01}p_{23} \\
C &= (p_{02}p_{03}+p_{12}p_{31})(\cos\theta-1)-p_{01}p_{23}\sin\theta \\
\theta &= \text{is the angle defining the cross-ratio of the complex, which is } e^{i\theta}
\end{align*}
\]
\( p_i \) are the coordinates of a real line referred to the real reference tetrahedron. The cross-ratio of a real line wrt the semi-imaginary tetrahedron is

\[
e^i = \frac{[p_{02} + ip_{12}] [p_{03} + ip_{31}]}{[p_{02} - ip_{12}] [p_{03} - ip_{31}]} \quad (11)
\]

Clearly the linear complexes \( ap_{01} + bp_{23} = 0 \) are trivially self-polar.

The real self-polar reguli given by (6) require \( q_{12} = Bp_{12} - Cp_{02} = dq_{03} = dBp_{03} + dCp_{31} \) so \( Bdp_{03} - Cp_{02} = dBp_{03} + dcCp_{02} \) requiring \( dc = -1 \), giving

\[
\{ p_{01} p_{02} p_{03} - dp_{03} p_{02} / d \ bp_{01} \} \quad (12)
\]

The self-polar quadric is \( bx_0^2 - bx_1^2 + x_2^2 / d - dx_3^2 = 0 \), so if we vary \( b \) the resulting congruence is hyperbolic with directrices \( \{0, d, \pm 1, \pm d, 1, 0\} \). If we vary \( d \) then \( x_2^2 + d_1 d_2 x_3^2 = 0 \) for common points. As \( d \) may be of either sign there are two sets of disjoint self-polar surfaces, and every member of one meets all members of the other. We thus have two elliptic congruences one for each sign of \( d \). Note that \( S_1 + \mu S_2 = 0 \) for two members \( S_1 \) and \( S_2 \) of (12) is not self-polar as \( -\mu d \) and \( \mu / d \) do not satisfy (12) other than for \( \mu = \pm 1 \).

Significant path curves arise with this type of tetrahedron, and the tangents to a path curve, forming a developable surface, all have the same cross-ratio wrt the invariant tetrahedron and thus belong to a tetrahedral complex. It can be shown that the tangents to the path curves on an axially symmetrical egg form a (2,2) congruence.

The polar figures of path curve developable surfaces which belong to co-singular complexes are not in general path curves but spiral ruled surfaces. The polars of the latter types of surface are again spiral ruled surfaces.
Fully-Imaginary Tetrahedron

For a “fully-imaginary” tetrahedron, which only has two skew lines as real elements, polar lines are given by

\[ q_{01} = A p_{01} \]
\[ q_{02} = B p_{02} + C p_{12} \]
\[ q_{03} = B p_{03} + C p_{31} \]
\[ q_{12} = B p_{12} + C p_{02} \]
\[ q_{31} = B p_{31} + C p_{03} \]
\[ q_{23} = A p_{23} \]

where

\[ A = (k^2-1)Q + 2(k^2+1)p_{01} p_{23} \]
\[ B = -4k p_{01} p_{23} \]
\[ C = (k-1)^2 Q + 2(k^2-1)p_{01} p_{23} \]
\[ Q = p_{02}^2 + p_{03}^2 + p_{12}^2 + p_{31}^2 \]

The cross-ratio \( k \) is real as for this type of tetrahedron real lines possess real cross-ratios.

The real self-polar quadrics require \( q_{12} = B p_{12} + C p_{02} = d q_{03} = d B p_{03} + d C p_{31} \), so \( B d p_{03} + C p_{02} = d B p_{03} + d C p_{02} \) and thus \( d c = 1 \), giving \{ \( p_{01} \ p_{02} \ p_{03} \ p_{03} \ p_{02} / d \ b p_{01} \} \). The self-polar quadric is \( b x_0^2 + b x_1^2 + x_2^2 / d + d x_3^2 = 0 \), so taking \( b \) as variable gives elliptic congruences with elliptic cross-sections in the horizontal plane \( (0 \ 0 \ 1 \ 0) \). For varying \( d \) we require \( x_2^2 - d_1 d_2 x_3^2 = 0 \) for common points, giving two hyperbolic congruences.
Bi-Correlation Transforms

Figure 9

Another approach to the whole subject is possible as follows. Let $S_1$ be the centre of a bundle of lines and $B_1$ the centre of a bundle of planes, and $C_1$ be a correlation between them so that every line $b$ in $S_1$ has a corresponding plane $\beta$ in $B_1$. Similarly, let $S_2$ and $B_2$ also be the centres of two similar bundles related by a correlation $C_2$. Then a point $P$ in space contains two lines $b_1$ and $b_2$ in $S_1$ and $S_2$ respectively to which correspond two planes $\beta_1$ and $\beta_2$ in $B_1$ and $B_2$ under $C_1$ and $C_2$. $\beta_1$ and $\beta_2$ meet in a line $p$ which is the image of $P$. We refer to this imaging process as a bi-correlation transform $\Gamma$. Thus the $\infty^3$ points of space are imaged by $\infty^3$ lines i.e. a line complex. In general this is a tetrahedral complex, but may degenerate. The tetrahedron is that with $S_1$, $S_2$, $B_1$, and $B_2$ as vertices, and the degenerate cases arise when special elements such as the line $S_1S_2$ correspond to planes in the line $B_1B_2$. That the faces of the tetrahedron are total planes is readily shown, and the vertices are total points. An essential feature is that $\Gamma$ has an inverse $\Gamma^{-1}$ which is also a bi-correlation transform, and incidences are preserved by both transforms (e.g. if two points lie on a line $p$ then the two lines imaging them lie on the regulus imaging $p$).

Congruences are studied as the transforms of surfaces, and reguli and flat pencils as transforms of lines. Many theorems and enumerative results are readily obtained by this method (once the singular elements are categorised) which are quite difficult to obtain algebraically.

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1 Long after developing this idea I found that Jessop describes another representation by points due to Klein and Nöther. It is not based on a pair of correlations, though.
As an example indicating the type of argument used, consider the congruence $C$ imaging a plane $\alpha$. To find its order we must see how many lines it has in a point $P$. The inverse image of $P$ is a line $p$ which intersects $\alpha$ in one point in the general case, so the order of the congruence is 1. Now consider a general plane $\mu$ inversely imaged by a collineation between the bundles in $S_1$ and $S_2$. The latter intersect $\alpha$ in a collineation of points which in general has three self-corresponding points imaged by three lines of $C$ lying in $\mu$, so $C$ is a (1,3) congruence.

As another example consider the intersection of a tetrahedral complex $K$ with a (1,3) congruence $C$. The latter is not an image of a plane since it does not lie in $K$ and hence contains lines that are not images of points. It will be inversely imaged by a collineation between the bundles in $S_1$ and $S_2$, and it is a standard result (S&K) that the lines in the bundles which do intersect generate a twisted cubic. The latter is imaged by the lines common to $K$ and $C$, and having $\infty^1$ lines it is a regulus.
Chaos

We saw in (3) that the cross-ratio of $p_{jk}$ wrt the reference tetrahedron is $-(p_{02}p_{31})/(p_{03}p_{12})$ so that the cross-ratio of the line $q_{jk}$ polar to $p_{jk}$ is, using (4),

$$\lambda' = \frac{-Bp_{02}Bp_{31}}{Cp_{03}Cp_{12}} = -\left(\frac{B}{C}\right)^2 \lambda$$

so from (F4)

$$\lambda' = \frac{(k^2 - 2k + \lambda)^2}{(k^2 - 2k \lambda + \lambda)^2} \lambda$$

(14)

If we start with a line $p_1$ with cross-ratio $\lambda$ then the cross-ratio of its polar is given by (14), and if we then take the result and substitute it for $\lambda$ on the right hand side of (14) we get the cross-ratio of a third line and so on i.e. we obtain the cross-ratios of the polar descendents of $p_1$. Depending on the value of $k$ the sequence of values of those cross-ratios is often chaotic since (14) is of the third degree, so that the polar descendents then move chaotically around the surface of their self-polar regulus. This is typical of many chaotic systems where chaotic behaviour occurs within a well-defined global domain.

Complete figures in cosingular complexes may thus be repeatedly transformed and follow a chaotic sequence e.g. a regulus or a congruence.

See Ref. Web page (sub page Volatile) for illustrations of this.

Forces

Polarity wrt the linear complex in association with the null system relates to physical forces (Struik). All the pairs of forces equivalent to a given pair of skew forces are given by the pairs of polar lines.
Since the tetrahedral complex arises naturally in the study of path curves it seems an interesting possibility suggested by Lawrence Edwards to the author that such a complex may have some relationship to life. A possible source of strain lies in the fact that the polar descendents of a line wrt to two cosingular complexes differ, although they lie in the same regulus. Where in Nature we see two different systems of path curves e.g. on many flower buds, pine cones, pineapples etc. we may suspect that two cosingular tetrahedral complexes are involved. So far the work of the author on counterspace has led to the view (unpublished) that matter is “frozen stress” i.e. stress that cannot relieve itself without some external action. Matter is woven light (c.f. Steiner), and light is “woven” into a CSI in this way, noting the great dependency plants have on light. Einstein's e=mc^2 expresses the (very large) ratio involved. The two distinct cosingular complexes above may engender such “frozen stress” in the living realm, matter arising as the bud grows between the two complexes.

Also in the work on counterspace the quadratic complex of tangents to a quadric surface may have application as there is a sphere called the valence sphere which represents the present time, for which such a complex would represent timeless transformations. This relates to quantum mechanics (Thomas), chemical action and time (unpublished results).
Appendix A
We will now prove that the lines \{p_{01} p_{02} dp_{03} cp_{02} bp_{01}\} lie on the quadric
\[bcdx_0^2+bx_1^2+cx_2^2+dx_3^2 = 0.\]

First of all we need to know the coordinates of the point \(x_j\) in which a line \(p_{jk}\) meets a plane \(l_j\). Suppose \(p_{jk}\) is the join of the points \(y_j\) and \(z_j\) so that a general point on \(p_{jk}\) is \(y_j + \lambda z_j\). Then for the point of intersection we require
\[\Sigma (y_j + \lambda z_j) l_j = 0.\]

Thus
\[\lambda = -(\Sigma y_j l_j)/(\Sigma z_j l_j).\]

Then \(x_0 = y_0 - z_0 (\Sigma y_j l_j)/(\Sigma z_j l_j)\), or \(x_0 = y_0 (\Sigma z_j l_j) - z_0 (\Sigma y_j l_j)\) since \((\Sigma z_j l_j)\) will be a common denominator for all the \(x_j\). Thus
\[x_0 = y_0 z_0 l_0 + y_0 z_1 l_1 + y_0 z_2 l_2 + y_0 z_3 l_3 - z_0 y_1 l_1 - z_0 y_2 l_2 - z_0 y_3 l_3 = p_{01} l_1 + p_{02} l_2 + p_{03} l_3,\]

and similarly for the other coordinates, giving
\[x_0 = p_{01} l_1 + p_{02} l_2 + p_{03} l_3 \quad x_1 = -p_{01} l_0 + p_{12} l_2 - p_{31} l_3 \quad x_2 = -p_{02} l_0 - p_{12} l_1 + p_{23} l_3 \quad x_3 = -p_{03} l_0 + p_{31} l_1 - p_{23} l_2 \]

Thus \(p_{jk}\) meets the plane \(\{1 0 0 0\}\) in the point \(y_j = \{0 -p_{01} -p_{02} -p_{03}\}\) and the plane \(\{0 1 0 0\}\) in \(z_j = \{p_{01} 0 -dp_{03} cp_{02}\}\) for a line of the regulus, so the line \(p_{jk}\) is
\[y_j + \lambda z_j = \{-\lambda p_{01}, p_{01}, p_{02} + \lambda dp_{03}, p_{03} - \lambda cp_{02}\} = \{x_0, x_1, x_2, x_3\}\]

Hence
\[x_2 = p_{02} + \lambda dp_{03}\]
and
\[x_3 = p_{03} - \lambda cp_{02}\]

Solving these for \(p_{02}\) and \(p_{03}\) gives
\[p_{02} = \frac{x_2 - \lambda dx_3}{1 + \lambda^2 c d}; \quad p_{03} = \frac{x_3 + \lambda cx_2}{1 + \lambda^2 c d}\]

The Plücker line condition for \(p_{jk}\) gives \(bp_{01}^2 + cp_{02}^2 + dp_{03}^2 = 0\), and substituting for the \(p_{jk}\) gives
\[b(1 + \lambda^2 cd)^2 x_1^2 + c(x_2 - \lambda dx_3)^2 + d(x_3 + \lambda cx_2)^2 = 0\]

noting that \(p_{01} = x_1\) from (A2).
Simplifying yields

\[ b(1+\lambda^2cd)x_1^2 + cx_2^2 + dx_3^2 = 0 \]

Noting that \( \lambda = -x_0/x_1 \) from (A2) this becomes

\[ bcdx_0^2 + bx_1^2 + cx_2^2 + dx_3^2 = 0 \]  \hspace{1cm} (A3)

which is the equation of the quadric containing the points \( x_j \) on the lines of the regulus, as was to be proved. Since it has no cross-product terms the reference tetrahedron is self-polar with respect to it.
Appendix B

We show that the polar of a line \( p \) intersecting an edge of \( T \) also intersects that edge, and the polar of a plane \( \alpha \) in an edge of \( T \) is the harmonic conjugate of \( \alpha \) wrt the tetrahedron.
Equations (4) and (5) are used below.

Given a line \( p_{ij} \) intersecting an edge of \( T \), one of its coordinates is zero, say \( p_{23} \) for example if it intersects \( e_{ij} = \{1 \ 0 \ 0 \ 0 \ 0 \} \).
The Plücker line condition gives \( p_{02}p_{31} = -p_{03}p_{12} \).
The Plücker line condition for its polar line is \( B^2p_{02}p_{31} + C^2p_{03}p_{12} = 0 \) whence \( B^2 = C^2 \).
Thus \( B = \pm C \). From (5) \( B = C \) implies that \( p_{02}p_{31} = p_{03}p_{12} \), which is false. So generally \( B = -C \). Thus the polar line is

\[
\{Ap_{01}/B \ p_{02} \ -p_{03} \ -p_{12} \ p_{31} \ 0\}
\]

which meets the edge \( e_{ij} \) in a point harmonically conjugate to where \( p_{ij} \) meets that edge wrt the two vertices of the tetrahedron on \( e_{ij} \).

From (A1) \( p_{ij} \) meets the plane \( \{1 \ 0 \ 0 \ 0 \} \) in \( \{0 \ -p_{01} \ -p_{02} \ -p_{03}\} \) and its polar meets that plane in \( \{0 \ -Ap_{01}/B \ -p_{02} \ p_{03}\} \) i.e. the two lines harmonically separate the planes \( \{0 \ 0 \ 1 \ 0\} \) and \( \{0 \ 0 \ 0 \ 1\} \). Hence the polar of the plane \( \alpha = (p_{ij}, e_{ij}) \) is the harmonic conjugate of \( \alpha \) wrt the tetrahedron.
Appendix C

We will prove von Staudt's Theorem that if a line $p$ intersects the faces of a tetrahedron in a range with cross-ratio $k$, then the axial pencil in $p$ subtended by the vertices has the same cross-ratio.

The tetrahedron $TUVW$ is shown with a transversal $ORC$ meeting the faces $TUW$, $TVW$, $UVW$ and $TUV$ at $C$, $R$, $O$ and $B$ respectively. Each intercept is joined to its opposite vertex by the lines shown. The plane $CVO$ intercepts $UW$ in $Y$, and $CY$ meets $TW$ in $X$ (since $C$, $Y$ and $TW$ all lie in the plane $TUW$). $VX$ is the line common to the planes $CVO$ and $TVW$, and as $CO$ lies in $CVO$ and meets $TVW$ in $R$, $R$ must lie on $VX$ i.e. $VRX$ are collinear. This fact will be assumed in the next diagram.
The axial pencil in the transversal CROB determined by the vertices TUVW is projected onto the plane TUW as follows: the planes in T U W give the lines CT CU CW respectively, and the plane in V gives the line CX. Thus the axial pencil BC(TU,VW) is in perspective with the pencil C(TZXW) which is in perspective with the range TZXW.

Now consider the axial pencil UV(TZXW): it is in perspective with C(TZXW), and also UV(TZXW)=UV(BCRO), so the cross-ratio (BC,RO) = UV(BC,RO) = UV(TZ,XW)=(TZ,XW)=C(TZ,XW)=BC(TU,VW) i.e. (BC,RO)=BC(TU,VW), as was to be proved.
**Appendix D**

We derive the Plücker coordinates from plane coordinates, and the dual of (A1). If the points $x_j$ and $y_j$ lie in the plane $u_j$ then

\[
\begin{align*}
\sum_{i=0}^{3} x_i u_i &= 0 \\
\sum_{i=0}^{3} y_i u_i &= 0
\end{align*}
\]

Eliminating $u_0$ gives

\[p_{01} u_1 + p_{02} u_2 + p_{03} u_3 = 0\]

Similarly if $x_j$ and $y_j$ lie in $v_j$

\[p_{01} v_1 + p_{02} v_2 + p_{03} v_3 = 0\]

Eliminating $p_{02}$ gives

\[(u_1 v_2 - u_2 v_1) p_{01} + (u_3 v_2 - u_2 v_3) p_{03} = 0\]

i.e.

\[
\frac{\pi_{23}}{\pi_{12}} = \frac{p_{01}}{p_{03}}
\]

where the $\pi_{ij}$ are based on plane coordinates. Similar results are obtained for other pairs of ratios, so that

\[p_{01}:p_{02}:p_{03}:p_{12}:p_{23} = \pi_{23}:\pi_{31}:\pi_{12}:\pi_{03}:\pi_{02}:\pi_{01} \quad (D1)\]

Thus the dual procedure of deriving line coordinates from two planes in the line gives the Plücker coordinates of the line as

\[\{\pi_{23} \pi_{31} \pi_{12} \pi_{03} \pi_{02} \pi_{01}\} \quad (D2)\]

If the coordinates used to derive (A1) are plane coordinates then $p_{01}$ will be replaced by $\pi_{01}$ etc. in (A1), so (D1) must be used to obtain the dual of (A1), which is thus

\[
\begin{align*}
\ell_0 &= p_{23} x_1 + p_{31} x_2 + p_{12} x_3 \\
\ell_1 &= -p_{23} x_0 + p_{03} x_2 - p_{02} x_3 \\
\ell_2 &= -p_{31} x_0 - p_{03} x_1 + p_{01} x_3 \\
\ell_3 &= -p_{12} x_0 + p_{02} x_1 - p_{01} x_2
\end{align*}
\]

(D3)

Giving the coordinates of the plane common to a point and a line.
Appendix E

We derive here the general coordinates of a line lying on a given *self-polar* quadric with cross-ratio $\lambda$. We may rewrite (6) as \{p_{01}/p_{03}, p_{02}/p_{03}, 1, d, cp_{02}/p_{03}, bp_{01}/p_{03}\} and then substitute for $p_{02}/p_{03}$ from (3), after writing $dp_{03}$ for $p_{12}$ and $bp_{02}$ for $p_{31}$ which gives

$$\{p_{01}/p_{03}, \pm \sqrt{d\lambda/c}, 1, d, \pm c \sqrt{-d\lambda/c}, bp_{01}/p_{03}\}$$

If we now apply the Plücker condition (1) to this we find that

$$p_{01}/p_{03} = \pm \sqrt{d(\lambda - 1)/b}$$

giving

$$\pm \sqrt{cd(\lambda - 1)}, \pm \sqrt{-bd\lambda}, \sqrt{bc}, d\sqrt{bc}, \pm c \sqrt{-bd\lambda}, \pm b \sqrt{cd(\lambda - 1)} \quad (D1)$$

This is in fact the general expression for the coordinates of a line with cross-ratio $\lambda$ belonging to a self-polar regulus determined by $b$ $c$ and $d$. The possibility of imaginary values arises from the fact that not all cross-ratios are possible on a particular regulus. We have three distinct cases:

1. **c and d have the same sign.**
   In this case the term $\sqrt{bc}$ is imaginary, so $\sqrt{cd(\lambda - 1)}$ is imaginary if $\lambda<1$, and $\sqrt{-bd\lambda}$ is imaginary if $\lambda<0$. Thus all terms are imaginary if $\lambda<0$, giving us real lines.

2. **b and d have the same sign.**
   Again $\sqrt{bc}$ is imaginary, and we require $\lambda>1$ for real lines.

3. **b and c have the same sign.**
   This requires $0<\lambda<1$.

For a congruence with variable $d$ but fixed $b$ and $c$, for example, the lines common to two of its reguli are given by the Plücker line condition:

$$2bc(\lambda - 1)\sqrt{d_1 d_2} - 2bc \lambda \sqrt{d_1 d_2} + bc (d_1 + d_2) = 0$$

which is satisfied by $b=0$ or $c=0$ for the parabolic congruences, and for the other congruences with neither zero we may re-express it as
\[ 2(\lambda - 1) \sqrt{\frac{d_2}{d_1}} + 2\lambda \sqrt{\frac{d_2}{d_1}} + 1 = 0 \]

showing that for a given pair of values of \( d_1 \) and \( d_2 \), which must be of the same sign, one value of \( \lambda \) is obtained giving one common line.

If we fix \( d_1 \) and vary \( d_2 \) we obtain all the lines of the regulus given by \( d_1 \), i.e. a complete self-polar regulus as part of the congruence.

There are \( \infty^1 \) pairs of values of \( d_1 \) and \( d_2 \) for a given value of \( \lambda \) and thus the congruence contains the \( \infty^1 \) lines of a regulus in each cosingular complex, for in effect we vary \( d \) for a given \( \lambda \). For a directrix \( q_{0i} \), (3) gives

\[ \sqrt{c d (\lambda - 1)}(q_{23} + b q_{01}) + \sqrt{-b d \lambda (q_{31} + c q_{02})} + q_{12} + d q_{03} = 0 \]

The lines \( \{q_{01}, q_{02}, 0, -cq_{02}, -bq_{01}\} \) satisfy this for all values of \( d \) and \( \lambda \), being the two lines \( \{\pm\sqrt{-b/c}, 1, 0, 0, -c, \mp\sqrt{-b/c}\} \), which are the directrices of the congruence.

For the elliptic congruences, \( b \) and \( c \) are of the same sign and the Plücker line condition fails for these directrices. However if we take \( q_{12} = q_{03} = 0 \) then \( d \) cancels and we get the condition \( b q_{01}^2 - q_{02} q_{31} - k|q_{01} q_{31} + c q_{02} q_{02}| = 0 \) after substituting for \( q_{23} \) to satisfy the Plücker condition, where \( k = \sqrt{|-b\lambda|/c(\lambda - 1)} \). For a given \( \lambda \) this gives a non-self-polar regulus of lines, so again the intersection of the congruence with the cosingular complex is a regulus. \( k \) varies with \( \lambda \) so there is no real directrix for all the lines of the complex.
Appendix F

We give here the derivation of equations (4) and (5). Referring to figure 6 and using (D3) for the plane common to the vertex \{1 0 0 0\} and the line \(x = p_{ij}\) whose polar we seek, the coordinates of the plane \(\alpha\) are

\[\alpha = (0, p_{23}, p_{31}, p_{12})\]

using the suffix 1 to indicate it is for the plane \((x, V_1)\). If the line \(k_1\) intersects the faces \{0 0 0 1\} \{0 0 1 0\} and \{0 1 0 0\} of the tetrahedron in \(A\) \(B\) and \(C\) respectively, the cross-ratio \((K_1 C, BA)\) equals that of the complex i.e. \(k\). We will find the coordinates of \(K_1\) using this. The plane \(\alpha\) intersects \(V_1 = \{1 0 0 0\}\) in the line \(\{p_{23}, p_{31}, p_{12}, 0, 0, 0\}\), and the point \(A\) is the intersection of this with the face \{0 0 0 1\}, and thus using (A1) it is \(\{0, -p_{31}, p_{23}, 0\}\).

Similarly \(B = \{0, p_{12}, 0, -p_{23}\}\) and \(C = \{0, 0, -p_{12}, p_{31}\}\). Expressing the line \(k_1\) as \(A + \lambda B\), the parameters of the four points \(K_1, C, B,\) and \(A\) are

\[\lambda_0 = \text{parameter of } K_1, \text{ to be calculated},\]
\[\lambda_1 = \text{parameter of } C = p_{31}/p_{12}\]
\[\lambda_2 = \text{parameter of } B = \infty\]
\[\lambda_3 = \text{parameter of } A = 0\]

and thus

\[k = (K_1 C, BA) = \frac{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_3)}{(\lambda_0 - \lambda_3)(\lambda_1 - \lambda_2)} = \frac{(\lambda_0 - \lambda_2)\left(p_{31}/p_{12} - 0\right)}{(\lambda_0 - \lambda_3)\left(p_{31}/p_{12} - \infty\right)}\]

Thus \(\lambda_0 = p_{31}/(p_{12}k)\), and \(K_1 = A + \lambda_0 B = \{0, p_{31}(1/k - 1), p_{23}, -p_{23}p_{31}/(p_{12}k)\}\). Then the line joining \(K_1\) and \(V_1\) is

\[\{p_{12}p_{31}(1-k), kp_{12}p_{23}, -p_{23}p_{31}, 0, 0, 0\}\]

Similarly the lines \(K_2 V_2, K_3 V_3\) and \(K_4 V_4\) are

\[\{p_{02}p_{03}(1-k), 0, 0, -p_{02}p_{23}, kp_{03}p_{23}, 0\}\]
\[\{0, p_{01}p_{03}k, 0, -p_{01}p_{31}, 0, p_{03}p_{31}(1-k)\}\]
\[\{0, 0, -p_{01}p_{02}, 0, p_{01}p_{12}k, p_{02}p_{12}(1-k)\}\].
Now we must find the two transversals of these four lines, which are \(x = p_{ij}\) and its polar \(q_{ij}\). A transversal \(y_{ij}\) must satisfy the Plücker intersection condition (3) for all four lines, which gives the matrix equation

\[
\begin{bmatrix}
 p_{12}p_{31}(1-k) & k p_{12}p_{23} & -p_{23}p_{31} & 0 \\
p_{02}p_{03}(1-k) & 0 & 0 & -p_{02}p_{23} \\
 0 & k p_{01}p_{03} & 0 & -p_{01}p_{31} \\
 0 & 0 & -p_{01}p_{02} & 0
\end{bmatrix}
\begin{bmatrix}
y_{23} \\
y_{31} \\
y_{12} \\
y_{03}
\end{bmatrix}
= \begin{bmatrix}
 0 & 0 \\
-k p_{03}p_{23} & 0 \\
 0 & -p_{03}p_{31}(1-k) \\
-k p_{01}p_{12} & -p_{02}p_{12}(1-k)
\end{bmatrix}
\begin{bmatrix}
y_{02} \\
y_{01}
\end{bmatrix}
\]

If this equation is expressed as \(M y' = -N y''\), then we invert \(M\) and calculate \(M^{-1}N\) to give

\[
\begin{bmatrix}
y_{23} \\
y_{31} \\
y_{12} \\
y_{03}
\end{bmatrix}
= \begin{bmatrix}
\alpha_{00} & \alpha_{10} \\
\alpha_{01} & \alpha_{11} \\
\alpha_{02} & \alpha_{12} \\
\alpha_{03} & \alpha_{13}
\end{bmatrix}
\begin{bmatrix}
y_{02} \\
y_{01}
\end{bmatrix}
\]

say, \((F1)\)

The Plücker line condition (2) requires

\[
0 = y_{01}y_{23} + y_{02}y_{31} + y_{03}y_{12}
= y_{01}(\alpha_{00}y_{02} + \alpha_{10}y_{01}) + y_{02}(\alpha_{01}y_{02} + \alpha_{11}y_{01}) + (\alpha_{03}y_{02} + \alpha_{13}y_{01})(\alpha_{02}y_{02} + \alpha_{12}y_{01})
\]

This is a quadratic equation in \(\frac{y_{01}}{y_{02}}\) for which the product of the roots is

\[
\frac{\alpha_{01} + \alpha_{02} + \alpha_{03}}{\alpha_{10} + \alpha_{12} + \alpha_{13}} \quad (F2)
\]

We find \(q_{01}/q_{02}\) from this by dividing it by \(p_{01}/p_{02}\) since we know that the latter is a root since \(p_{ij}\) is one of the transversals. The inverse of \(M\) is
which is readily checked by multiplying it by $M$ to obtain the identity matrix $I$. Since $N$ is

$$
\begin{bmatrix}
0 & 0 \\
-k p_{03}p_{23} & 0 \\
0 & -p_{03}p_{31}(1-k) \\
-k p_{01} p_{12} & -p_{02} p_{12}(1-k)
\end{bmatrix}
$$

$M^{-1}N$ is given by

$$
\begin{bmatrix}
\alpha_{00} & \alpha_{10} \\
\alpha_{01} & \alpha_{11} \\
\alpha_{02} & \alpha_{12} \\
\alpha_{03} & \alpha_{13}
\end{bmatrix}
\begin{bmatrix}
0 & p_{23} \\
\frac{p_{31}}{p_{01}} & 0 \\
\frac{p_{02}}{p_{02}} & \frac{(1-k)p_{12}}{p_{01}} \\
\frac{k p_{03}}{p_{02}} & \frac{(1-k)p_{03}}{p_{01}}
\end{bmatrix}
$$

and thus the product of the roots (F2) is
\[ p_{31}/p_{02} + \left( k^2 p_{12}/p_{02} \right) p_{03}/p_{02} \]
\[ /p_{23}/p_{01} + |1-k|^2 p_{12}/p_{03}/p_{01} \]

Dividing by \( p_{01}/p_{02} \) gives

\[ \frac{p_{01} p_{02} p_{31} + k^2 p_{01} p_{03} p_{12}}{p_{01} p_{02} p_{23} + (1-k)^2 p_{02} p_{03} p_{12}} = q_{01} \]
\[ \frac{p_{02}}{p_{02}} \rho_{02} \]

i.e.

\[ q_{01} = p_{01}(p_{02} p_{31} + k^2 p_{03} p_{12}) \]
\[ q_{02} = p_{02}(p_{01} p_{23} + (1-k)^2 p_{03} p_{12}) \]

The other coordinates of \( q_{ij} \) are found from (F1) and (F3):

\[ q_{23} = \begin{bmatrix} \alpha_{00} & \alpha_{10} \end{bmatrix} \begin{bmatrix} q_{02} \\ q_{01} \end{bmatrix} = p_{23}(p_{02} p_{31} + k^2 p_{03} p_{12}) \]

and similarly

\[ q_{31} = p_{31}(p_{01} p_{23} + (1-k)^2 p_{03} p_{12}) \]
\[ q_{12} = p_{12}(k p_{01} p_{23} + (1-k)p_{02} p_{31} + k(1-k)p_{03} p_{12}) \]
\[ q_{03} = p_{03}(k p_{01} p_{23} + (1-k)p_{02} p_{31} + k(1-k)p_{03} p_{12}) \]

Applying the Plücker line condition (2) we have

\[ kp_{01} p_{23} + (1-k)p_{02} p_{31} + k(1-k)p_{03} p_{12} = k^2 p_{01} p_{23} + (1-k)^2 p_{02} p_{31} \]

finally giving us

\[ q_{01} = A p_{01} \]
\[ q_{02} = B p_{02} \]
\[ q_{03} = C p_{03} \]
\[ q_{12} = C p_{12} \]
\[ q_{31} = B p_{31} \]
\[ q_{23} = A p_{23} \]

where

\[ A = p_{02} p_{31} + k^2 p_{03} p_{12} \]
\[ B = p_{01} p_{23} + (1-k)^2 p_{03} p_{12} \]
\[ C = k^2 p_{01} p_{23} + (1-k)^2 p_{02} p_{31} \]
We may find alternative expressions for A, B and C which can be useful. If we divide the equations for A, B and C by \( p_{03}p_{12} \), then if \( \lambda \) is the cross-ratio of the original line \( p_{jk} \) we substitute for \( (p_{02}p_{31})/(p_{03}p_{12}) \) from (3) with \( \lambda \) in place of \( k \). We also note using (3) and (1) that \( p_{01}p_{23}/p_{03}p_{12} = \lambda - 1 \) and thus obtain

\[
\begin{align*}
A &= k^2 - \lambda \\
B &= k^2 - 2k + \lambda \\
C &= -(k^2 - 2k\lambda + \lambda)
\end{align*}
\]  

(F4)

Note that if \( \lambda = k \) for a line of the complex then \( A = B = C \) and the line is self-polar.
References


Web page: www.nct.anth.org.uk