PIVOT TRANSFORMS

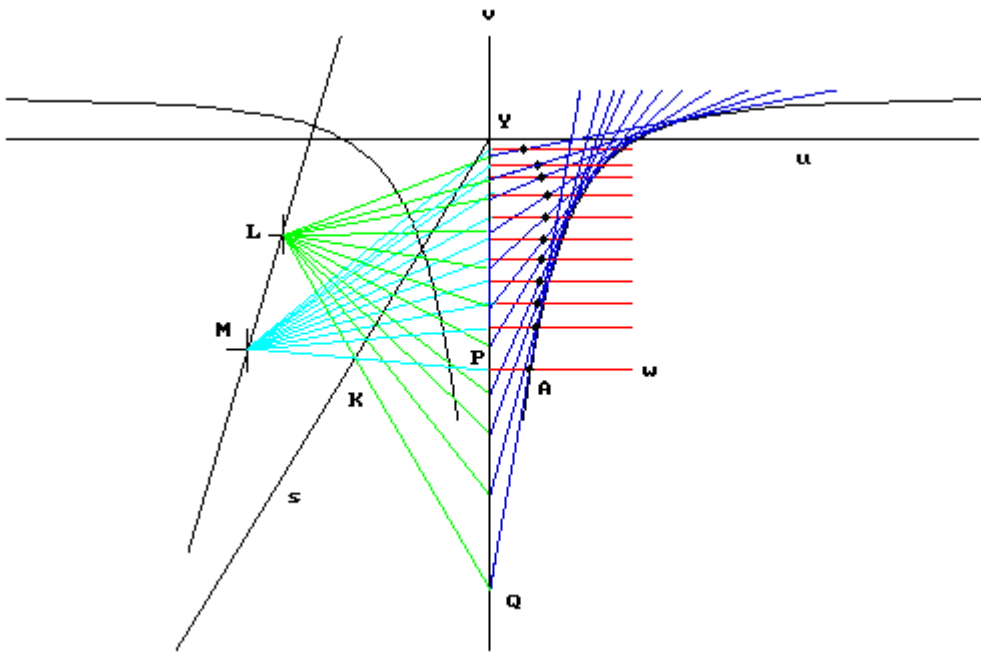
ANNEX 3

CONSTRUCTION OF PIVOT TRANSFORMS

The formula relating h and h' is

$$h=\frac{k_2 h'}{k_2 h'+k_1(1-h')} \tag{1}$$

This is a linear relation between h and h' for given k_j and hence represents a projectivity between Q and P, where Q is the vertical intercept of the pivoting plane with the axis, and P the intercept with the vertical axis of the horizontal plane in which the pivot point lies. Thus to construct a profile it suffices to set up this projectivity. It is illustrated below:



The projectivity is defined by the two centres of projection L and M and the auxiliary line v. v must pass through one invariant point (Y here) and LM through the other. Then if the vortex shown is being transformed, we start with a tangent plane intersecting the axis in a point such as Q. We project Q into P as shown, and the horizontal plane through P meets the tangent plane in a line w on which the pivot point A lies. The above diagram assumes this is the nearest point to P, which is only true if $(\theta_2-\theta_1)=(2n+1)\pi$ (or in practice $n\pi$). Otherwise A lies on the line such that PA is at an angle $(\theta_2-\theta_1)$ to w. Since this angle is constant for a given transformation the rotation amounts to a scaling by $\cos(\theta_2-\theta_1)$, which may either be ignored as it affects all the radii PA alike, or absorbed into the projectivity if absolute dimensions are significant for the particular application.

The projectivity is arrived at as follows. The angles α_2 and α_3 (see main text) give k_1 and k_2. Choose any intercept such as Q, so that YQ=-h' (in the above case where Q is below the lower invariant plane). Calculate h from (1) above, and mark P such that YP=|h|. Choose any line v through Y and any line LM

through the upper invariant point. Select a position for L, draw QL meeting v in K; then M is the point where PK meets LM.

If we wish to start from particular values of λ and ε for the bud transformation, then we use the following formulae to calculate α_2 and α_3:

$$\alpha_2 = -\tan^{-1}\left(\frac{\lambda+1}{2\varepsilon\lambda}\right)$$
$$\alpha_3 = \tan^{-1}\left(\frac{\lambda+1}{2\varepsilon}\right) \qquad (2)$$

It will be recalled that k_1 and k_2 are then given by

$$k_1 = \sin^2\alpha_3$$

$$k_2 = \sin^2\alpha_2$$

or from elementary trigonometry

$$k_1 = \frac{(\lambda+1)^2}{4\varepsilon^2+(\lambda+1)^2}$$
$$k_2 = \frac{(\lambda+1)^2}{4\lambda^2\varepsilon^2+(\lambda+1)^2} \qquad (3)$$

so that using (1) as before we set up the projectivity between h and h':

$$h = \frac{h'\left[4\varepsilon^2+(1+\lambda)^2\right]}{4h'\varepsilon^2(1-\lambda^2)+(1+\lambda)^2+4\varepsilon^2\lambda^2} \qquad (4)$$

To relate this to Reference 1, our $\alpha_2 = -\theta_1$
$\alpha_3 = \theta_2$

and k_1 and k_2 are there defined as

$$k_1 = \frac{\sin(\theta_1+\theta_2)}{\sin^2\theta_1}$$
$$k_2 = \frac{\sin(\theta_1+\theta_2)}{\sin^2\theta_2}$$

It will be seen that substitution of these in (1) above gives the same result as using our definitions since $\sin(\theta_1+\theta_2)$ cancels.

Proof of Equations (2)

In (9) and (10) in the main text we saw that

$$\lambda = -\frac{\log\phi - \log\Lambda_2}{\log\phi - \log\Lambda_3}$$

and

$$\varepsilon = \frac{\log\Lambda_2 - \log\Lambda_3}{2\theta}$$

Thus

$$\frac{\lambda+1}{2\varepsilon} = \frac{\theta}{\log\phi - \log\Lambda_3}$$

which when compared with (0) in the main text verifies the equation for α_3.

Dividing our last expression by λ then gives

$$\frac{\lambda+1}{2\,\varepsilon\,\lambda} = \frac{-\theta}{\log\phi - \log\Lambda_2}$$

which verifies the equation for α_2.

Addendum

Solving (1) for the double points by setting h=h', we see that one solution is clearly h=h'=0, and the other is h=h'=1. These are the two points where the lines v and LM respectively intersect the vertical axis (remembering that h and h' are fractional heights). They are independent of the form being transformed, depending only upon the bud transform.

If a photograph of an actual bud with its seed-pod is being analysed it is possible to estimate by eye where the seed pod profile meets the axis at a point . This corresponds to $h'=-\infty$, for which $h=k_2/(k_2-k_1)=q$, say Z. Using (3) we can then find ε (knowing λ from the bud profile):

$$q = \frac{\dfrac{(1+\lambda)^2}{\left(4\lambda^2\varepsilon^2+(1+\lambda)^2\right)}}{\dfrac{(1+\lambda)^2}{\left(4\lambda^2\varepsilon^2+(1+\lambda)^2\right)} - \dfrac{(1+\lambda)^2}{\left(4\varepsilon^2+(1+\lambda)^2\right)}} = \frac{4\varepsilon^2+(1+\lambda)^2}{4\varepsilon^2\left(1-\lambda^2\right)}$$

Hence

$$\varepsilon = \frac{1+\lambda}{2\sqrt{q(1-\lambda^2)-1}} \qquad (5)$$

taking the positive value. For $\lambda>1$ for a bud this should be real as $q<0$ (as a fractional height). Then from (3) we have

$$k_1 = \frac{q\left(1-\lambda^2\right)-1}{q\left(1-\lambda^2\right)}$$
$$k_2 = \frac{q\left(1-\lambda^2\right)-1}{(q-1)\left(1-\lambda^2\right)} \tag{6}$$

(1) then simplifies to

$$h = \frac{qh'}{h'+q-1} \tag{7}$$

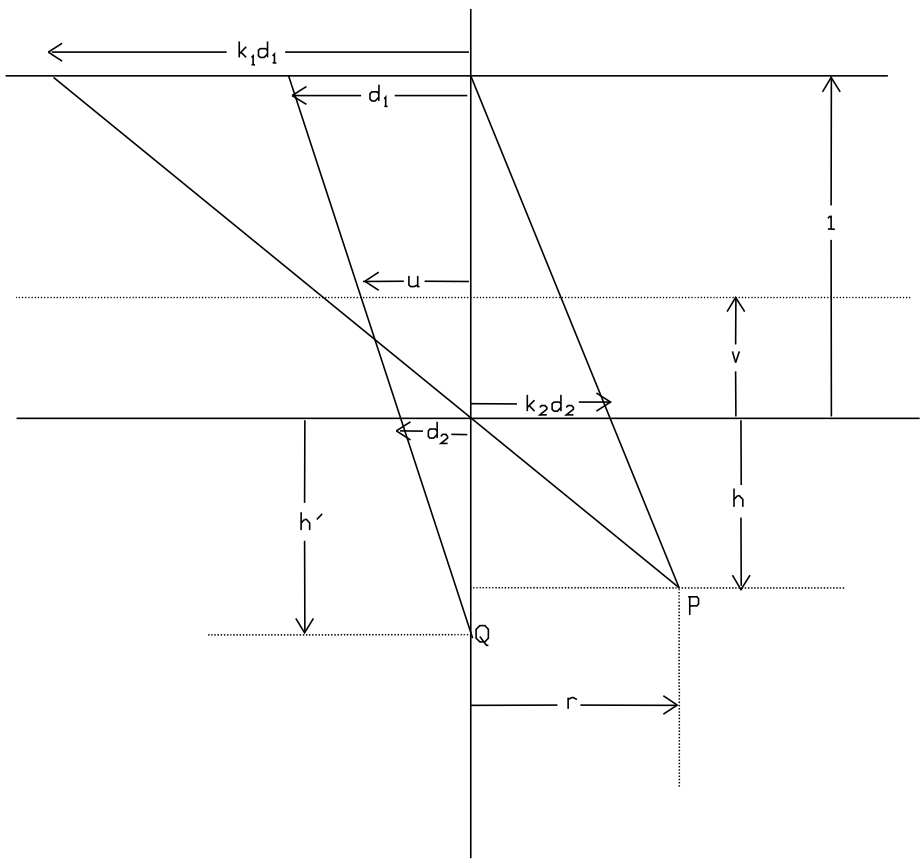
If we convert from fractional to measured heights (using the measured distance between the double points on the bud), we multiply h, h' and q by a scaling factor σ to give

$$H = \frac{QH'}{H'+Q-\sigma} \tag{8}$$

For r we have from (8) of the main document

$$r = r_2 hk$$
$$where$$
$$k = \frac{\sin(\alpha_2+\alpha_3)}{\sin^2\alpha_2}$$

If we do not know α_1 and α_2 then this does not help. So we choose three points on the profile of the gynoecium and calculate λ_c of the cosmic vortex from them. The following diagram helps to clarify this:



This shows a plane orthogonal to the paper meeting the central axis in Q and the top and bottom planes of the bud transformation at distances d_1 and d_2 respectively from the axis. d_1k_1 and d_2k_2 are then the distances from the axis of the points through which two lines one from each invariant point meet in the pivot point P. All dimensions are fractional and positive upwards, as the distance between the bud planes is taken as 1. For a given h' we want to find r. If we know the vortex lambda (λ_c) then the distance u of the tangent plane from the axis is related to the height H=v-h' of Q from the vortex invariant plane (shown dashed) by the equation

$$u = u_0 \left(\frac{H}{H_0} \right)^{\frac{\lambda_c}{1+\lambda_c}} \tag{9}$$

of the required vortex, where (u_0,H_0) is for a known reference point on the gynoecium. Then d_1 and d_2 are easily found by similar triangles, whence k_1d_1 and k_2d_2 give P and hence r.

We must thus determine λ_c and v in order to use this to plot the gynoecium. We take three points on it, one for (r_0,y_0) and thence (u_0,H_0) and two others to give (u_1,H_1) and (u_2,H_2). For the correct value of v and λ_c we would have

$$\frac{\log\left(\frac{u_1}{u_0}\right)}{\log\left(\frac{H_1}{H_0}\right)} = \frac{\log\left(\frac{u_2}{u_0}\right)}{\log\left(\frac{H_2}{H_0}\right)}$$

so we use bisection on v to make the difference zero (as H=v-h') which gives us v. Then λ_c follows immediately from (9).

Thus we can find the transform without knowing ε if we can estimate Q.

To construct it without calculation, draw any line v as above and a line through the other invariant point on which L and M will lie. Then select L and draw a vertical line through it to meet v in a point W. Join the point Z to W and extend it to meet v in M. The transform is now set up.

The tangent lines to the vortex can then be constructed by reversing the sequence of the original construction: select a point Q on the axis, join it to L meeting v in K, join MK meeting the axis in P, draw a horizontal line through P to meet the seed pod profile in A, and finally join AQ to give the tangent.