CENTRIFUGAL PUMP BASED ON PATH CURVES

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Introduction

This paper derives the method used to calculate the operating conditions of a centrifugal pump based on the use of path curves for the vanes and vertical profile, on the assumption the stream lines will then be path curves. The configuration is as shown in Figure 1.

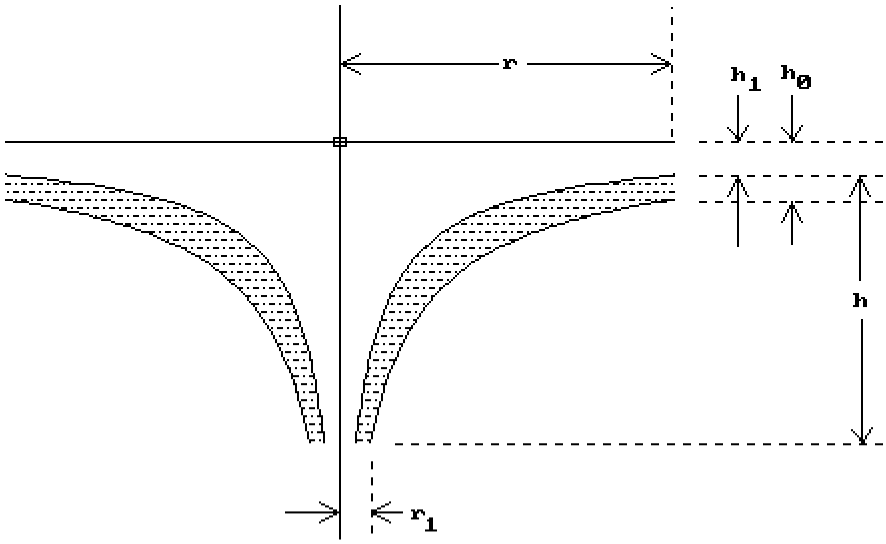


FIGURE 1

Centrifugal Force

Figure 2 shows the vertical cross-section profile of a streamline, which is a vortical path curve with the vertex at infinity. Two such vortices of rotation with different radii form the inner and outer sheaths of the pump (Figure 1). The profile path curve is defined by a parameter λ which is typically –2, and the path curves defining the vanes are determined by two parameters λ (as above) and ε which controls the three-dimensional spiralling of the curve (Reference 1). The angle α which is the constant angle between the tangents and radii of a logarithmic spiral, may be used instead of ε in this case where the vertex is at infinity. It is given by

$$\tan\alpha = \frac{1+\lambda}{2\varepsilon} \qquad (1)$$

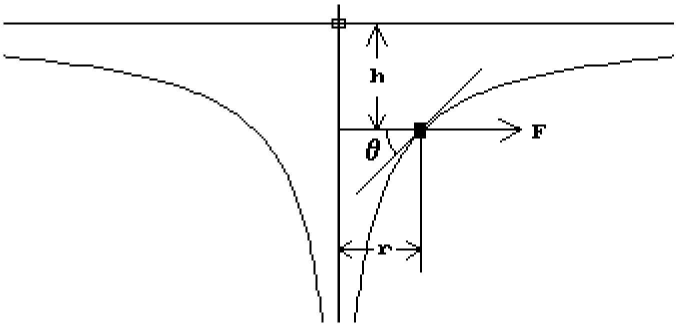


FIGURE 2

An element of volume of water is shown on a vertical profile, on which a centrifugal force is acting due to the rotation. Assuming the streamlines are the (λ, α) path curves, we resolve the force along the tangent to that curve. For a profile path curve with one invariant point at infinity the equation relating the height and radius is (Reference 1)

$$h = kr^{1+\lambda} \quad where \quad k = \frac{h_0}{r_0^{1+\lambda}} \tag{2}$$

h_0 and r_0 are the height and radius at a suitable reference point on the streamline. The height is measured relative to the accessible invariant plane, and k varies with the streamlines.

If the density of the water is ρ and the angular velocity is ω then the centrifugal force on an elemental annular ring of volume δV at a radius r is $\rho \delta V r \omega^2$. It is shown in Appendix A that to resolve this along the tangents to the path curves passing through the ring we multiply by

$$\frac{1}{\sqrt{1+tan^2\theta+tan^2\alpha}} \tag{3}$$

where θ is as shown in Figure 2 and α is a constant as defined previously. From (2) we have

$$tan\,\theta = \frac{dh}{dr} = (1+\lambda)kr^{\lambda} = \frac{(1+\lambda)h}{r} \tag{4}$$

so the driving force on an element of volume, using (3) and (4), is

$$\frac{\rho \delta V r \omega^2}{\sqrt{1+tan^2\alpha+tan^2\theta}} = \frac{\rho \delta V r \omega^2 cos\,\alpha}{\sqrt{1+\left\{\frac{(1+\lambda)h\,cos\,\alpha}{r}\right\}^2}} \tag{5}$$

The total driving force along all the streamlines is the double integral of (5). We integrate first wrt h, in which case we have a cylinder of fixed r intersecting the inner and outer vortical sheaths at heights given by $h_1 = k_1 r^{1+\lambda}$ and $h_2 = k_2 r^{1+\lambda}$ which are the limits of the integration. After these limits have been inserted we integrate wrt r with limits r_1 and r_2 which are the greatest and least radii of the outer vortical sheath. This excludes a small portion of water at the base of the pump where the height integral would be invalid as the lower limit would go outside the physical dimensions of the pump.

Since $\delta V = 2\pi r \delta h \delta r$, the integral of (5) is

$$2\pi\rho\omega^2 \cos\alpha \int_{r_1}^{r_2} r^2 \left[\int_{h_1}^{h_2} \frac{1}{\sqrt{1+\frac{\{(1+\lambda)h\cos\alpha\}^2}{r^2}}} dh \right] dr \tag{6}$$

Although the first integral may be solved explicitly, the second may not, so it is simplest to expand as a series and then integrate term-by-term. This gives

$$2\pi\rho\omega^2 \cos\alpha \int_{r_1}^{r_2} r^2 dr \int_{h_1}^{h_2} \left[1 + \sum_{n=1}^{\infty} {}_{-1/2}C_n \left(\frac{(1+\lambda)h\cos\alpha}{r} \right)^{2n} \right] dh$$

=

$$2\pi\rho\omega^2 \cos\alpha \int_{r_1}^{r_2} r^2 dr \left[h + \sum_{n=1}^{\infty} {}_{-1/2}C_n \left(\frac{(1+\lambda)\cos\alpha}{r} \right)^{2n} \frac{h^{2n+1}}{2n+1} \right]_{k_1 r^{1+\lambda}}^{k_2 r^{1+\lambda}}$$

where we insert the limits using (2), k_1 and k_2 being for the inner and outer vortical sheaths.
The final integration is thus

$$2\pi\rho\omega^2 \cos\alpha \int_{r_1}^{r_2} \left[(k_2-k_1) r^{3+\lambda} + \sum_{n=1}^{\infty} {}_{-1/2}C_n \left((1+\lambda)\cos\alpha \right)^{2n} \frac{r^{(2n+1)(1+\lambda)-2n+2}}{2n+1} \left(k_2^{2n+1} - k_1^{2n+1} \right) \right] dr$$

=

$$2\pi\rho\omega^2 \cos\alpha \left[\sum_{n=0}^{\infty} {}_{-1/2}C_n \left((1+\lambda)\cos\alpha \right)^{2n} \frac{r^{(2n+1)\lambda+4}}{(2n+1)((2n+1)\lambda+4)} \left(k_2^{2n+1} - k_1^{2n+1} \right) \right]_{r_1}^{r_2} \tag{7}$$

To compute this practically, define $T_n = {}_{-1/2}C_n[(1+\lambda)^2 \cos^2\alpha\ k^2 r^{2\lambda}]^n$

so $T_n = T_{n-1}[1/(2n)-1]\ \{(1+\lambda)\cos\alpha\ kr^{\lambda}\}^2$ for $n > 0$ (8)

and $T_0 = kr^{\lambda+4}$

The term added to the sum is thus $T_n/(2n+1)((2n+1)\lambda+4)$.
Complete the summing when, say, the n'th term has dropped to 0.1%, and then multiply by $2\pi\rho\omega^2\cos\alpha$.

If the dimensions of the pump are (see Figure 1):

- h = the height of the pump
- h_0 = the distance of the top of the outer vortical sheath from the invariant plane
- h_1 = the distance of the inner vortical sheath from that plane (which is the top of the pump)
- r = the maximum radius of the pump (the same for both sheaths)

then the limits are given by

$$k_2 = h_0/r^{1+\lambda} \qquad \text{as } (r,h_0) \text{ lies on the outer sheath}$$

$$r_2 = r \tag{9}$$

$$r_1 = \{(h+h_1)/k_2\}^{1/(1+\lambda)} \qquad \text{as } (r_1,h+h_1) \text{ lies on the outer sheath}$$

$$k_1 = h_1/r^{1+\lambda} \qquad \text{as } (r,h_1) \text{ lies on the inner sheath}$$

Two observations must be made:

1. If $\lambda < -4$ the second term of (7) is negative and no driving force is available.
   If $\lambda = -4$ the second term is infinite, so we must have $-1 > \lambda > -4$
2. The series fails to converge if $\{(1+\lambda)\cos\alpha\, kr^{\lambda}\}^2 > 1$.
   This occurs for r_1 for both values of k as e.g. $k_2 r_1^{\lambda} = (h+h_1)/r_1 > 1$ in most practical cases.
   However, if instead we use limits $r = 0$ to r_2 then (7) is zero for $r = 0$ and the convergence problem is evaded. In effect we then only use the limit pairs r_2,k_2 and r_2,k_1 and thus evaluate the force for two pumps, one between the axis and the outer sheath, the other between the axis and the inner sheath, and subtract. This follows because ignoring the other two limit pairs amounts to solving (6) twice, first for the limits $h=0$ to $k_2 r^{\lambda}$ and $r=0$ to r_2, and then for the limits $h=0$ to $k_1 r^{\lambda}$ and $r=0$ to r_2. In fact this is a more accurate result than initially envisaged as all the water in the pump is included (c.f. comments after equation (5)). But it is not completely accurate as it now includes the water that would be in the pump were h larger, because the height limit is a function of r. As this extra space is almost vertical it generates virtually no driving force but does have weight, and may be taken informally to allow for losses.

<u>Calculation of Opposing force due to Weight</u>

The component of centrifugal force along the streamline is opposed by the weight, which must also be resolved (Figure 3).

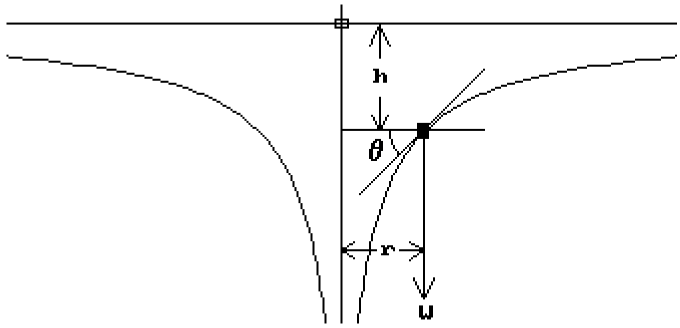


FIGURE 3

The weight of an annular element of volume is $2\pi r \rho g \delta r \delta h$. It is shown in Appendix A that to resolve this along the path curve we must multiply by

$$\frac{1}{\sqrt{1+cot^2\theta\, sec^2\alpha}} \tag{10}$$

which in conjunction with (4) gives

$$\frac{1}{\sqrt{1+\frac{r^2 sec^2\alpha}{(1+\lambda)^2 h^2}}} = \frac{(1+\lambda)cos\,\alpha\;\, h/r}{\sqrt{1+\frac{(1+\lambda)^2 h^2 cos^2\alpha}{r^2}}} \tag{11}$$

so the integral is

$$2\pi\rho\, g\int_{r_1}^{r_2} r\; dr\int_{h_1}^{h_2}\left[\frac{(1+\lambda)h\, cos\,\alpha \big/ r}{\sqrt{1+(1+\lambda)^2 h^2 cos^2\alpha \big/ r^2}}\right] dh \tag{12}$$

The integration wrt h may be solved directly as follows

$$2\pi\rho\, g\int_{r_1}^{r_2}\frac{r^2}{(1+\lambda)cos\,\alpha}\, dr\int_{h_1}^{h_2}\left[\frac{(1+\lambda)^2 h\, cos^2\alpha \big/ r^2}{\sqrt{1+(1+\lambda)^2 h^2 cos^2\alpha \big/ r^2}}\right] dh$$

giving

$$\frac{2\pi\rho\, g\, sec\,\alpha}{1+\lambda}\int_{r_1}^{r_2} r^2\left[\sqrt{1+(1+\lambda)^2 h^2 cos^2\alpha \big/ r^2}\right]_{k_1 r^{1+\lambda}}^{k_2 r^{1+\lambda}} dr$$

but this has the defect that when $\alpha = 90°$, sec$\alpha = \infty$ but in fact the result should be zero as $\varepsilon = 0$ and thus the streamlines become horizontal circles with zero <u>resolved</u> weight. A similar problem arises if we expand the LHS of (11) as a series, because then all terms but the first contain sec$^2\alpha$. Therefore we expand (12) as a series to ensure correct results for all α:

$$2\pi\rho\, g(1+\lambda)cos\,\alpha\int_0^{r_2} dr\int_{k_1 r^{1+\lambda}}^{k_2 r^{1+\lambda}}\left[\sum_{n=0}^{\infty}{}_{-1/2}C_n\left\{\frac{(1+\lambda)cos\,\alpha}{r}\right\}^{2n} h^{2n+1}\right] dh \tag{13}$$

$$= \quad 2\pi\rho\, g(1+\lambda)cos\,\alpha\int_0^{r_2}\left[\sum_{n=0}^{\infty}{}_{-1/2}C_n\left\{\frac{(1+\lambda)cos\,\alpha}{r}\right\}^{2n}\frac{h^{2n+2}}{2n+2}\right]_{k_1 r^{1+\lambda}}^{k_2 r^{1+\lambda}} dr$$

$$= \qquad 2\pi\rho g(1+\lambda)\cos\alpha \int_0^{r_2} \left[\sum_{n=0}^{\infty} {}_{-1/2}C_n \{(1+\lambda)\cos\alpha\}^{2n} \frac{\left(k_2^{2(n+1)} - k_1^{2(n+1)}\right)}{2n+2} r^{2(n+1)(1+\lambda)-2n} \right] dr$$

$$= \qquad 2\pi\rho g(1+\lambda)\cos\alpha \left[\sum_{n=0}^{\infty} {}_{-1/2}C_n \{(1+\lambda)\cos\alpha\}^{2n} \frac{\left(k_2^{2(n+1)} - k_1^{2(n+1)}\right)}{2n+2} \frac{r_2^{2(n+1)\lambda+3}}{2(n+1)\lambda+3} \right] \tag{14}$$

provided $\lambda \neq -3/2$ when n = 0. For larger n we have $\lambda > -1$ for zero denominator, which is in any case invalid as the pump would then be upside down.

For practical computation, let

$$T_n = {}_{-1/2}C_n\{(1+\lambda)^2 k^2 r^{2\lambda} \cos^2\alpha\}^n = T_{n-1}[1/(2n)-1]\{(1+\lambda) k r^{\lambda} \cos\alpha\}^2 \quad n>0 \tag{15}$$

The term added to the sum is thus $\dfrac{T_n}{2(n+1)\{2(n+1)\lambda+3\}}$, and $T_0 = k^2 r^{2\lambda+3}$.

The limits are as given in (9), but as before we only use the limit pairs r_2,k_2 and r_2,k_1.

Calculation of Flowrate

The nett force developed = F = (resolved centrifugal force – resolved weight) acts on the surface area A_E of the outlet. The dynamic pressure at the outlet is thus

$$F/A_E \tag{16}$$

The discharge $Q = v_E A_E = v_I A_I$ (17)
where A_I = inlet area
v_E = exit or outlet velocity
v_I = inlet velocity

Applying Bernoulli's equation (Reference 2 page 312) we get

$$\frac{1}{2}v_E^2 + \frac{Fg}{A_E\gamma} = \frac{1}{2}v_I^2 = \frac{1}{2}v_E^2\frac{A_E^2}{A_I^2} \tag{18}$$

where γ = specific weight of water, say at 20°C

Thus $\frac{1}{2}v_E^2\left\{\frac{A_E^2}{A_I^2} - 1\right\} = \frac{Fg}{A_E\gamma}$

so

$$v_E = \sqrt{\frac{2\,Fg}{\gamma\,A_E\left(A_E^2/A_I^2 - 1\right)}} \tag{19}$$

From this the discharge Q may be calculated using (17), and the power required for steady flow, ignoring losses, is

$$\text{Power} = Q\gamma\ h \tag{20}$$

where h is the head.

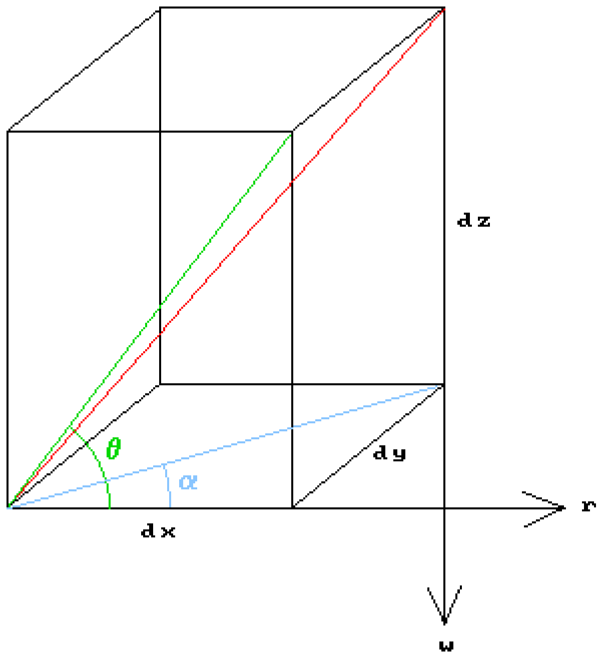
References

1. *Practical Path Curve Calculations*, N C Thomas (Available via www.nct.anth.org.uk/people)

2. *Fluid Mechanics*, Streeter & Wylie, ISBN 0-07-062232-9

APPENDIX A

We wish to resolve the force vector along the tangent to the path curve, which is simple when a special path curve is involved for which equation (1) applies. The figure shows an elemental parallelepiped at a point on the curve, with the curve as a local straight line segment (red) passing from corner to corner:



α is the angle between the tangent and radius for the logarithmic spiral component of the path curve, and θ is the angle between the tangent to the profile component of the path curve and the horizontal.

If dx lies in the radial direction then we resolve the force along r in the direction of the path curve by multiplying by a factor

$$\frac{dx}{\sqrt{dx^2+dy^2+dz^2}} = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2}} = \frac{1}{\sqrt{1+\tan^2\alpha+\tan^2\theta}}$$

which (as a check) reduces to cosθ when α = 0 for a vertical curve, or to cos α when θ = 0 for a horizontal curve.

To resolve the weight acting vertically down, suppose it to be w acting as shown so that the factor is

$$\frac{dz}{\sqrt{dx^2+dy^2+dz^2}} = \frac{1}{\sqrt{1+\left(\frac{dx}{dz}\right)^2+\left(\frac{dy}{dx}\right)^2\left(\frac{dx}{dz}\right)^2}} = \frac{1}{\sqrt{1+\cot^2\theta+\tan^2\alpha\cot^2\theta}} = \frac{1}{\sqrt{1+\cot^2\theta\sec^2\alpha}}$$

which reduces to sinθ when α = 0 for a vertical curve, or to zero when θ = 0 for a horizontal curve.